

Component reduction in $\mathcal{N} = 2$ supergravity: the vector, tensor, and vector-tensor multiplets

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ABSTRACT: Recent advances in curved $\mathcal{N} = 2$ superspace methods have rendered the component reduction of superspace actions more feasible than in the past. In this paper, we consider models involving both vector and tensor multiplets coupled to supergravity and demonstrate explicitly how component actions may be efficiently obtained. In addition, tensor multiplets coupled to conformal supergravity are considered directly within projective superspace, where their formulation is most natural. We then demonstrate how the inverse procedure – the lifting of component results to superspace – can simplify the analysis of complicated multiplets. We address the off-shell $\mathcal{N} = 2$ vector-tensor multiplet coupled to conformal supergravity with a central charge and demonstrate explicitly how its constraints and Lagrangian can be written in a simpler way using superfields.

Contents

1	Introduction	2
2	Superspace geometry and conformal supergravity	4
2.1	Conformal superspace	4
2.2	One-form geometry of the abelian vector multiplet	7
2.3	Two-form geometry of the tensor multiplet	8
2.4	SU(2) superspace geometry	9
3	Component actions from superspace	12
3.1	General component chiral actions from superspace	14
3.2	The abelian vector multiplet in components	16
3.3	Abelian vector multiplet action	17
3.3.1	The superconformal case	17
3.3.2	The general case	18
3.4	The tensor multiplet in components	22
3.5	Tensor multiplet action	23
4	Projective superspace and component reductions	25
4.1	Formalism of projective superspace	25
4.2	Component reductions in projective superspace	28
4.3	Abelian vector and tensor multiplets in projective superspace	29
4.4	Component evaluation	29
4.4.1	Globally superconformal action	30
4.4.2	Construction of the abelian vector multiplet one-form	32
4.4.3	Locally superconformal action	36
5	Lifting superspace results from components	40
5.1	Vector-tensor multiplet in supergravity	40
5.2	Reinterpreting the constraints	42
5.3	Action formulation for the vector-tensor multiplet	45
5.4	Comparison with existing superspace approaches	48
6	Discussion	48
A	Notations and conventions	51
B	Solution to constraints for the tensor multiplet	52

C	Component results from superspace	55
C.1	Conformal superspace in components	55
C.2	SU(2) superspace in components	56

1 Introduction

Supergravity-matter model building has attracted substantial interest ever since the first models for supergravity were found over thirty years ago. Much of this interest has been due to its intimate relationship with complex geometry. For globally supersymmetric $\mathcal{N} = 1$ theories, the Kähler structure of the target space is most apparent when cast in the language of superfields in superspace. The same may be said of the Hodge-Kähler geometry of locally supersymmetric theories. The most general matter Lagrangians, when written in terms of chiral superfields in superspace, naturally lead to the required geometries with the Lagrangian playing the role of the generating function for the target space geometry. This is due to the relative simplicity of $\mathcal{N} = 1$ superspace, its corresponding action principle, and the relative ease of moving between $\mathcal{N} = 1$ superspace and component actions. Both approaches are important. Superspace methods economically incorporate supersymmetry and provide generating formulations, while component methods are usually given directly by higher dimensional supergravity and string computations. Therefore, it is advantageous to be able to move deftly between them.

However, for the case of $\mathcal{N} = 2$ supersymmetric theories, especially locally supersymmetric theories, the relationship between superspace and component methods has remained distant. $\mathcal{N} = 2$ component methods, *e.g.* those based on the $\mathcal{N} = 2$ superconformal tensor calculus [1–3], are quite mature and have been used to describe off-shell vector and tensor multiplets and on-shell hypermultiplets and their couplings to conformal supergravity. However, they cannot describe off-shell hypermultiplets without a central charge and do not provide a generating formulation for the most general couplings.

In superspace, this last problem was overcome with the development of harmonic [4, 5] and projective [6–9] superspace, in which general supersymmetric models involving off-shell hypermultiplets could be constructed. These models naturally lead to the hyperkähler target space geometries required by rigid $\mathcal{N} = 2$ supersymmetry upon elimination of the infinite numbers of auxiliary fields. However, coupling such theories to supergravity remained problematic for some time, at least for projective superspace.¹

A recent advance in superspace has resolved this last problem. In 2008, a projective superspace version of general $\mathcal{N} = 2$ supergravity-matter couplings was constructed by Kuzenko, Lindström, Roček, and Tartaglino-Mazzucchelli [10].² This new formulation

¹Models in harmonic superspace can be coupled directly to the supergravity prepotentials [4, 5]. However, the relation between this and the differential geometric structure of superspace – that is, the full algebra of covariant derivatives – remains unclear.

²The construction was inspired by Kuzenko and Tartaglino-Mazzucchelli’s solution to five dimensional supergravity and its matter couplings, which also made use of projective superspace [11–13].

allows the most general couplings of off-shell $\mathcal{N} = 2$ matter to conformal supergravity and (at least in principle) provides a generating formalism to produce the most general on-shell supergravity-matter component actions.

This approach, like all conventional superspace methods, was based on a superspace geometry where superconformal transformations were not part of the structure group; rather, they were manifested as nonlinear super-Weyl transformations. For this reason, matching the component structure of these general actions to prior work involving the superconformal tensor calculus remained nontrivial. Recently, this issue has also been resolved by the construction of a curved $\mathcal{N} = 2$ superspace, which has been called conformal superspace [14], which possesses the full superconformal algebra in its structure group and reproduces exactly in its component structure the $\mathcal{N} = 2$ superconformal tensor calculus. Though the structure group is more complicated, the algebra of covariant derivatives is markedly simpler than more conventional superspace approaches. Since conformal superspace can be “degauged” to the conventional superspaces, it provides a useful bridge between existing superspace constructions and the superconformal tensor calculus.

The goal of this paper is to make use of both these recent advances in curved $\mathcal{N} = 2$ superspace techniques to bridge some of the gap which remains between superspace and component methods. The layout of the paper is as follows. In section 2, we briefly review the superspace geometry for conformal supergravity as well as for the one-form and two-form potentials corresponding to the vector and tensor multiplets. Section 3 is devoted to the component reduction of some simple superspace actions. Section 4 presents a component reduction in the projective superspace formulation of conformal supergravity, which is a novel calculation. We choose a relatively simple action for which we can easily verify the result.

In section 5, we reverse the process of component reduction and demonstrate how certain very complicated multiplets can simplify dramatically when recast in a superfield setting. The multiplet which we address is the off-shell vector-tensor multiplet [15] coupled to supergravity with a central charge, which received a great deal of interest in the late 1990s due to its appearance in $\mathcal{N} = 2$ supersymmetric vacua of heterotic string theory [16]. Although several off-shell formulations of this multiplet have been constructed in flat superspace before [17–24], the intricate off-shell formulation of Claus *et al.* [25–27], which is coupled to conformal supergravity with a central charge, has to our knowledge only been fully understood at the component level. We will show how the use of superfields can simplify a number of its features and demonstrate that it is equivalent to the superspace formulations of the vector-tensor multiplet coupled to supergravity recently considered in [28] as a curved generalization of previous results.³

In addition, we include several technical appendices. Appendix A summarizes the notations and conventions which we will be using throughout this paper. Appendix B gives the technical details of the solution to the Bianchi identities for the $\mathcal{N} = 2$ tensor multiplet in conformal superspace. Appendix C gives a number of useful identities and relations for the superspace geometries which we will be using.

³See also the alternative formulations of [29, 30] and [31, 32].

2 Superspace geometry and conformal supergravity

The Weyl multiplet of 4D $\mathcal{N} = 2$ conformal supergravity [1–3] can be realized in a number of different ways in superspace. The comprehensive analysis due to Howe [33] addressed the general case of \mathcal{N} -extended superspace (with $\mathcal{N} \leq 4$) and chose the structure group $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{U}(\mathcal{N})_{\mathrm{R}}$. The geometry uncovered there may be understood (at least for $\mathcal{N} \leq 2$) at the component level as conformal supergravity coupled to a real scalar supermultiplet of non-vanishing Weyl weight. This scalar supermultiplet is a compensator, and a change in the choice of compensator field manifests as a nonlinear super-Weyl transformation in superspace. When coupled to a superconformal action, the compensator becomes a pure gauge degree of freedom. A simpler version of $\mathcal{N} = 2$ conformal supergravity in superspace was constructed in [10] using the superspace geometry derived earlier by Grimm [34] using the structure group $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SU}(2)_{\mathrm{R}}$. This $\mathrm{SU}(2)$ superspace may be understood as a gauge-fixed version of $\mathrm{U}(2)$ superspace (*i.e.* Howe’s geometry for the case $\mathcal{N} = 2$) [35]. Recently, $\mathrm{SU}(2)$ superspace has proven useful in the construction of a projective superspace formulation of conformal supergravity [10].⁴

However, the superspace geometry which we will utilize most in this paper is neither $\mathrm{U}(2)$ nor $\mathrm{SU}(2)$ superspace, but a more general one which gauges not just $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{U}(2)_{\mathrm{R}}$ but the entire superconformal algebra $\mathrm{SU}(2, 2|2)$. This formulation, known as $\mathcal{N} = 2$ conformal superspace [14] has the advantage that it reduces *precisely* to conformal supergravity in components, without any additional compensating multiplet. A drawback (or feature) is that only superconformal actions can be described with it; but if the covariant derivatives are “degauged”, $\mathrm{U}(2)$ superspace results.

In this section we review $\mathcal{N} = 2$ conformal superspace [14], adapted to the notation used in [10]. We give a brief discussion of both the one-form (vector) and two-form (tensor) superspace geometries. Then we end with a discussion of the formulation of $\mathcal{N} = 2$ superspace due to Grimm (elaborated upon in [10]), explaining how it may be recovered from the formulation of [14] using the method of compensators.

2.1 Conformal superspace

We begin with a curved 4D $\mathcal{N} = 2$ superspace $\mathcal{M}^{4|8}$ parametrized by local bosonic (x) and fermionic ($\theta, \bar{\theta}$) coordinates $z^M = (x^m, \theta^\mu_\iota, \bar{\theta}^{\dot{\iota}}_\mu)$, where $m = 0, 1, \dots, 3$, $\mu = 1, 2$, $\dot{\mu} = 1, 2$ and $\iota = \underline{1}, \underline{2}$. The Grassmann variables θ^μ_ι and $\bar{\theta}^{\dot{\iota}}_\mu$ are related to each other by complex conjugation: $\overline{\theta^\mu_\iota} = \bar{\theta}^{\dot{\iota}\mu}$. The structure group is chosen to be $\mathrm{SU}(2, 2|2)$ and the covariant derivatives $\nabla_A = (\nabla_a, \nabla^i_\alpha, \bar{\nabla}^{\dot{\alpha}}_{\dot{i}})$ have the form

$$\begin{aligned} \nabla_A &= E_A + \frac{1}{2} \Omega_A^{ab} M_{ab} + \Phi_A^{ij} J_{ij} + i \Phi_A Y + B_A \mathbb{D} + \mathfrak{F}_A^B K_B \\ &= E_A + \Omega_A^{\beta\gamma} M_{\beta\gamma} + \bar{\Omega}_A^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}} + \Phi_A^{ij} J_{ij} + i \Phi_A Y + B_A \mathbb{D} + \mathfrak{F}_A^B K_B . \end{aligned} \quad (2.1)$$

Here $E_A = E_A^M(z) \partial_M$ is the supervielbein, with $\partial_M = \partial / \partial z^M$, $J_{kl} = J_{lk}$ are generators of the group $\mathrm{SU}(2)_{\mathrm{R}}$, M_{ab} are the Lorentz generators, Y is the generator of the chiral rotation

⁴This approach was later extended to $\mathrm{U}(2)$ superspace [35].

group $U(1)_R$, and $K^A = (K^a, S_i^\alpha, \bar{S}_{\dot{\alpha}}^i)$ are the special superconformal generators.⁵ The one-forms Ω_A^{bc} , Φ_A^{kl} , Φ_A , B_A and \mathfrak{F}_A^B are the corresponding connections. The conventions we use here differ in numerous ways from those used originally in [14]. For the most part, they follow the conventions of [36] and [10, 35] and are summarized in appendix A.

The Lorentz generators obey

$$\begin{aligned} [M_{ab}, M_{cd}] &= 2\eta_{c[a}M_{b]d} - 2\eta_{d[a}M_{b]c} , \quad [M_{ab}, \nabla_c] = 2\eta_{c[a}\nabla_{b]} , \\ [M_{ab}, \nabla_\alpha^i] &= (\sigma_{ab})_\alpha{}^\beta \nabla_\beta^i , \quad [M_{ab}, \bar{\nabla}_{\dot{\alpha}}^i] = (\tilde{\sigma}_{ab})_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\nabla}_{\dot{\beta}}^i . \end{aligned} \quad (2.2)$$

The $SU(2)_R$, $U(1)_R$ and dilatation generators obey

$$\begin{aligned} [J_{ij}, J_{kl}] &= -\varepsilon_{k(i}J_{j)l} - \varepsilon_{l(i}J_{j)k} , \quad [J_{ij}, \nabla_\alpha^k] = -\delta_{(i}^k \nabla_{\alpha j)} , \quad [J_{ij}, \bar{\nabla}_{\dot{k}}^{\dot{\alpha}}] = -\varepsilon_{k(i} \bar{\nabla}_{\dot{j})}^{\dot{\alpha}} , \\ [Y, \nabla_\alpha^i] &= \nabla_\alpha^i , \quad [Y, \bar{\nabla}_{\dot{\alpha}}^i] = -\bar{\nabla}_{\dot{\alpha}}^i , \\ [\mathbb{D}, \nabla_a] &= \nabla_a , \quad [\mathbb{D}, \nabla_\alpha^i] = \frac{1}{2}\nabla_\alpha^i , \quad [\mathbb{D}, \bar{\nabla}_{\dot{\alpha}}^i] = \frac{1}{2}\bar{\nabla}_{\dot{\alpha}}^i . \end{aligned} \quad (2.3)$$

The special superconformal generators K^A transform in the obvious way under Lorentz and $SU(2)_R$ rotations,

$$\begin{aligned} [M_{ab}, K_c] &= 2\eta_{c[a}K_{b]} , \quad [M_{ab}, S_i^\gamma] = -(\sigma_{ab})_\beta{}^\gamma S_i^\beta , \quad [M_{ab}, \bar{S}_{\dot{\gamma}}^i] = -(\tilde{\sigma}_{ab})^{\dot{\beta}}{}_{\dot{\gamma}} \bar{S}_{\dot{\beta}}^i , \\ [J_{ij}, S_k^\gamma] &= -\varepsilon_{k(i}S_{j)}^\gamma , \quad [J_{ij}, \bar{S}_{\dot{\gamma}}^k] = -\delta_{(i}^k \bar{S}_{\dot{j})}^{\dot{\gamma}} , \end{aligned} \quad (2.4)$$

while their transformation under $U(1)_R$ and dilatations is opposite that of ∇_A :

$$\begin{aligned} [Y, S_i^\alpha] &= -S_i^\alpha , \quad [Y, \bar{S}_{\dot{\alpha}}^i] = \bar{S}_{\dot{\alpha}}^i , \\ [\mathbb{D}, K_a] &= -K_a , \quad [\mathbb{D}, S_i^\alpha] = -\frac{1}{2}S_i^\alpha , \quad [\mathbb{D}, \bar{S}_{\dot{\alpha}}^i] = -\frac{1}{2}\bar{S}_{\dot{\alpha}}^i . \end{aligned} \quad (2.5)$$

Among themselves, the generators K^A obey the algebra

$$\{S_i^\alpha, \bar{S}_{\dot{\alpha}}^j\} = 2i\delta_i^j (\sigma^a)_{\dot{\alpha}}{}^\alpha K_a . \quad (2.6)$$

Finally, the algebra of K^A with ∇_B is given by

$$\begin{aligned} [K^a, \nabla_b] &= 2\delta_b^a \mathbb{D} + 2M^a{}_b , \\ \{S_i^\alpha, \nabla_\beta^j\} &= 2\delta_i^j \delta_\beta^\alpha \mathbb{D} - 4\delta_i^j M^\alpha{}_\beta - \delta_i^j \delta_\beta^\alpha Y + 4\delta_\beta^\alpha J^i{}_j , \\ \{\bar{S}_{\dot{\alpha}}^i, \bar{\nabla}_{\dot{\beta}}^j\} &= 2\delta_j^i \delta_{\dot{\alpha}}^{\dot{\beta}} \mathbb{D} + 4\delta_j^i \bar{M}_{\dot{\alpha}}{}^{\dot{\beta}} + \delta_j^i \delta_{\dot{\alpha}}^{\dot{\beta}} Y - 4\delta_{\dot{\alpha}}^{\dot{\beta}} J^i{}_j , \\ [K^a, \nabla_\beta^j] &= -i(\sigma^a)_{\beta}{}^{\dot{\beta}} \bar{S}_{\dot{\beta}}^j , \quad [K^a, \bar{\nabla}_{\dot{\beta}}^j] = -i(\sigma^a)^{\dot{\beta}}{}_\beta S_j^\beta , \\ [S_i^\alpha, \nabla_b] &= i(\sigma_b)^\alpha{}_{\dot{\beta}} \bar{\nabla}_{\dot{\beta}}^i , \quad [\bar{S}_{\dot{\alpha}}^i, \nabla_b] = i(\sigma_b)_{\dot{\alpha}}{}^\beta \nabla_\beta^i , \end{aligned} \quad (2.7)$$

where all other (anti-)commutations vanish.

⁵Following common usage, we will refer to K^a as the special conformal generator and S_i^α as the S -supersymmetry generator.

The covariant derivatives obey (anti-)commutation relations of the form

$$\begin{aligned} [\nabla_A, \nabla_B] &= T_{AB}^C \nabla_C + \frac{1}{2} R_{AB}^{cd} M_{cd} + R_{AB}^{kl} J_{kl} \\ &\quad + i R_{AB}(Y) Y + R_{AB}(\mathbb{D}) \mathbb{D} + R_{AB}^C K_C , \end{aligned} \quad (2.8)$$

where T_{AB}^C is the torsion, and R_{AB}^{cd} , R_{AB}^{kl} , $R_{AB}(Y)$, $R_{AB}(\mathbb{D})$ and R_{AB}^C are the curvatures. Some of the components of the torsion and curvature must be constrained. Following [14], the spinor derivative torsions and curvatures are chosen to obey

$$\{\nabla_\alpha^i, \nabla_\beta^j\} = -2\varepsilon^{ij} \varepsilon_{\alpha\beta} \bar{\mathcal{W}} , \quad \{\bar{\nabla}_i^{\dot{\alpha}}, \bar{\nabla}_j^{\dot{\beta}}\} = 2\varepsilon_{ij} \varepsilon^{\dot{\alpha}\dot{\beta}} \mathcal{W} , \quad \{\nabla_\alpha^i, \bar{\nabla}_j^{\dot{\beta}}\} = -2i\delta_j^i \nabla_\alpha^{\dot{\beta}} , \quad (2.9)$$

where \mathcal{W} is some operator valued in the superconformal algebra. In [14], it was shown how to constrain \mathcal{W} entirely in terms of a superfield $W_{\alpha\beta}$ so that the component structure reproduces $\mathcal{N} = 2$ conformal supergravity. In our notation, the constraints lead to

$$\{\nabla_\alpha^i, \nabla_\beta^j\} = 2\varepsilon^{ij} \varepsilon_{\alpha\beta} \bar{W}_{\dot{\gamma}\dot{\delta}} \bar{M}^{\dot{\gamma}\dot{\delta}} + \frac{1}{2} \varepsilon^{ij} \varepsilon_{\alpha\beta} \bar{\nabla}_{\dot{\gamma}k} \bar{W}^{\dot{\gamma}\dot{\delta}} \bar{S}_{\dot{\delta}}^k - \frac{1}{2} \varepsilon^{ij} \varepsilon_{\alpha\beta} \nabla_{\gamma\delta} \bar{W}^{\dot{\delta}}_{\dot{\gamma}} K^{\gamma\dot{\gamma}} , \quad (2.10a)$$

$$\{\bar{\nabla}_i^{\dot{\alpha}}, \bar{\nabla}_j^{\dot{\beta}}\} = -2\varepsilon_{ij} \varepsilon^{\dot{\alpha}\dot{\beta}} W^{\gamma\delta} M_{\gamma\delta} + \frac{1}{2} \varepsilon_{ij} \varepsilon^{\dot{\alpha}\dot{\beta}} \nabla^{\gamma k} W_{\gamma\delta} S_k^\delta - \frac{1}{2} \varepsilon_{ij} \varepsilon^{\dot{\alpha}\dot{\beta}} \nabla^{\gamma\dot{\gamma}} W_{\gamma}{}^\delta K_{\delta\dot{\gamma}} , \quad (2.10b)$$

$$\{\nabla_\alpha^i, \bar{\nabla}_j^{\dot{\beta}}\} = -2i\delta_j^i \nabla_\alpha^{\dot{\beta}} , \quad (2.10c)$$

$$\begin{aligned} [\nabla_{\alpha\dot{\alpha}}, \nabla_\beta^i] &= -i\varepsilon_{\alpha\beta} \bar{W}_{\dot{\alpha}\dot{\beta}} \bar{\nabla}^{\dot{\beta}i} - \frac{i}{2} \varepsilon_{\alpha\beta} \bar{\nabla}^{\dot{\beta}i} \bar{W}_{\dot{\alpha}\dot{\beta}} \mathbb{D} - \frac{i}{4} \varepsilon_{\alpha\beta} \bar{\nabla}^{\dot{\beta}i} \bar{W}_{\dot{\alpha}\dot{\beta}} Y + i\varepsilon_{\alpha\beta} \bar{\nabla}_j^{\dot{\beta}} \bar{W}_{\dot{\alpha}\dot{\beta}} J^{ij} \\ &\quad - i\varepsilon_{\alpha\beta} \bar{\nabla}_\beta^i \bar{W}_{\dot{\gamma}\dot{\alpha}} \bar{M}^{\dot{\beta}\dot{\gamma}} - \frac{i}{4} \varepsilon_{\alpha\beta} \bar{\nabla}_\alpha^i \bar{\nabla}_k^{\dot{\beta}} \bar{W}_{\dot{\beta}\dot{\gamma}} \bar{S}^{\dot{\gamma}k} + \frac{1}{2} \varepsilon_{\alpha\beta} \nabla^{\gamma\dot{\beta}} \bar{W}_{\dot{\alpha}\dot{\beta}} S_\gamma^i \\ &\quad + \frac{i}{4} \varepsilon_{\alpha\beta} \bar{\nabla}_\alpha^i \nabla_\gamma^{\dot{\gamma}} \bar{W}^{\dot{\beta}\dot{\gamma}} K_{\gamma\dot{\beta}} , \end{aligned} \quad (2.10d)$$

$$\begin{aligned} [\nabla_{\alpha\dot{\alpha}}, \bar{\nabla}_i^{\dot{\beta}}] &= i\delta_\alpha^{\dot{\beta}} W_{\alpha\beta} \nabla_i^\beta + \frac{i}{2} \delta_\alpha^{\dot{\beta}} \nabla_i^\beta W_{\alpha\beta} \mathbb{D} - \frac{i}{4} \delta_\alpha^{\dot{\beta}} \nabla_i^\beta W_{\alpha\beta} Y + i\delta_\alpha^{\dot{\beta}} \nabla^{\beta j} W_{\alpha\beta} J_{ij} \\ &\quad + i\delta_\alpha^{\dot{\beta}} \nabla_i^\beta W_\alpha^\gamma M_{\beta\gamma} + \frac{i}{4} \delta_\alpha^{\dot{\beta}} \nabla_{\alpha i} \nabla^{\beta j} W_{\beta}{}^\gamma S_{\gamma j} - \frac{1}{2} \delta_\alpha^{\dot{\beta}} \nabla_\gamma^\beta W_{\alpha\beta} \bar{S}_i^{\dot{\gamma}} \\ &\quad + \frac{i}{4} \delta_\alpha^{\dot{\beta}} \nabla_{\alpha i} \nabla_\gamma^{\dot{\gamma}} W_{\beta\gamma} K^{\beta\dot{\gamma}} . \end{aligned} \quad (2.10e)$$

The complex superfield $W_{\alpha\beta} = W_{\beta\alpha}$ and its complex conjugate $\bar{W}_{\dot{\alpha}\dot{\beta}} := \overline{W_{\alpha\beta}}$ are superconformally primary, $K_A W_{\alpha\beta} = 0$, and obey the additional constraints

$$\bar{\nabla}_i^{\dot{\alpha}} W_{\beta\gamma} = 0 , \quad \nabla_{\alpha\beta} W^{\alpha\beta} = \bar{\nabla}^{\dot{\alpha}\dot{\beta}} \bar{W}_{\dot{\alpha}\dot{\beta}} , \quad (2.11)$$

where we introduce the notation

$$\nabla_{\alpha\beta} := \nabla_{(\alpha}^k \nabla_{\beta)k} , \quad \bar{\nabla}^{\dot{\alpha}\dot{\beta}} := \bar{\nabla}_k^{(\dot{\alpha}} \bar{\nabla}^{\dot{\beta})k} . \quad (2.12)$$

Despite the appearance of the S -supersymmetry and special conformal K_a generators, the algebra of covariant derivatives (2.10) is significantly simpler to work with than the corresponding algebras of $SU(2)$ [10, 34] or $U(2)$ superspace [33, 35].

2.2 One-form geometry of the abelian vector multiplet

It is possible to introduce a one-form gauge potential, describing the $\mathcal{N} = 2$ vector multiplet, in the formulation of conformal supergravity presented in the previous section, as a generalization of the flat superspace solution [37]. We shall consider the abelian vector multiplet, since it will be used extensively in subsequent sections and its generalization to the non-abelian case is straightforward.

The field strength two-form F is given in terms of its one-form potential $V = dz^A V_A$ by $F = dV$, or equivalently,

$$F_{AB} = 2\nabla_{[A}V_{B]} - T_{AB}{}^C V_C . \quad (2.13)$$

Due to the existence of the one-form potential the field strength must satisfy the Bianchi identity

$$dF = 0 \implies \nabla_{[A}F_{BC]} - T_{[AB}{}^D F_{D|C]} = 0 . \quad (2.14)$$

At mass dimension-1 we impose the constraints

$$F_{\alpha\beta}^{ij} = -2\varepsilon^{ij}\varepsilon_{\alpha\beta}\bar{\mathcal{W}} , \quad F_i^{\dot{\alpha}\dot{\beta}}{}_j = 2\varepsilon_{ij}\varepsilon^{\dot{\alpha}\dot{\beta}}\mathcal{W} , \quad F_{\alpha j}^{i\dot{\beta}} = 0 , \quad (2.15)$$

where \mathcal{W} is a primary superfield with dimension 1 and U(1) weight -2 ,

$$K_A \mathcal{W} = 0 , \quad \mathbb{D} \mathcal{W} = \mathcal{W} , \quad Y \mathcal{W} = -2\mathcal{W} . \quad (2.16)$$

Then the Bianchi identities may be solved giving

$$F_{a\beta}^j = \frac{i}{2}(\sigma_a)_{\beta}{}^{\dot{\gamma}} \bar{\nabla}_{\dot{\gamma}}^j \bar{\mathcal{W}} , \quad F_{aj}^{\dot{\beta}} = -\frac{i}{2}(\sigma_a)_{\gamma}{}^{\dot{\beta}} \nabla_j^{\gamma} \mathcal{W} , \quad (2.17a)$$

$$F_{ab} = -\frac{1}{8}(\sigma_{ab})_{\alpha\beta}(\nabla^{\alpha\beta}\mathcal{W} + 4W^{\alpha\beta}\bar{\mathcal{W}}) + \frac{1}{8}(\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}}(\bar{\nabla}^{\dot{\alpha}\dot{\beta}}\bar{\mathcal{W}} + 4\bar{W}^{\dot{\alpha}\dot{\beta}}\mathcal{W}) . \quad (2.17b)$$

\mathcal{W} is further required to be a reduced chiral superfield,

$$\bar{\nabla}_{\dot{\alpha}}^i \mathcal{W} = 0 , \quad \nabla^{ij} \mathcal{W} = \bar{\nabla}^{ij} \bar{\mathcal{W}} . \quad (2.18)$$

Here we have introduced the notation

$$\nabla^{ij} := \nabla^{\gamma(i} \nabla^{j)}_{\gamma} , \quad \bar{\nabla}^{ij} := \bar{\nabla}_{\dot{\gamma}}^{(i} \bar{\nabla}^{j)}^{\dot{\gamma}} . \quad (2.19)$$

As a straightforward extension of the results found in [14] we can introduce a gauged central charge using the off-shell vector multiplet, generalizing the superspace formulation in [10]. First, we introduce a modified covariant derivative:

$$\nabla_A := \nabla_A + V_A \Delta , \quad (2.20)$$

where $V_A(z)$ is the gauge connection and Δ is a real central charge. Its (anti-)commutation relations are

$$\begin{aligned} [\nabla_A, \nabla_B] &= T_{AB}{}^C \nabla_C + \frac{1}{2} R_{AB}{}^{cd} M_{cd} + R_{AB}{}^{kl} J_{kl} \\ &\quad + i R_{AB}(Y) Y + R_{AB}(\mathbb{D}) \mathbb{D} + R_{AB}{}^C K_C + F_{AB} \Delta , \end{aligned} \quad (2.21)$$

where the torsion and curvature remain the same as in (2.8). The central charge abelian field strength F_{AB} obeys (2.15) and (2.17), with \mathcal{Z} (instead of \mathcal{W}) denoting the corresponding reduced chiral superfield, which is annihilated by the central charge.

2.3 Two-form geometry of the tensor multiplet

The $\mathcal{N} = 2$ tensor multiplet [38, 39] can be described by a two-form gauge potential in supergravity. Its formulation in $U(2)$ superspace was given in, *e.g.*, [40, 41]. The extension to conformal superspace is entirely straightforward and we summarize it briefly here.

The field strength three-form H is given in terms of its two-form gauge potential $B = \frac{1}{2}E^B E^A B_{AB}$ by

$$H = dB = \frac{1}{3!}E^C E^B E^A H_{ABC} , \quad H_{ABC} = 3\nabla_{[A}B_{BC]} - 3T_{[AB}{}^D B_{D|C]} . \quad (2.22)$$

The field strength remains invariant under gauge transformations $\delta B = dV$ with V a one-form gauge parameter. The existence of the gauge potential requires that the Bianchi identity

$$dH = 0 \implies \nabla_{[A}H_{BCD]} - \frac{3}{2}T_{[AB}{}^E H_{E|CD]} = 0 \quad (2.23)$$

be satisfied. As with the one-form, we must impose constraints. At mass dimension- $\frac{3}{2}$ they consist of

$$H_{\alpha\beta\gamma}^{ijk} = H_i^{\dot{\alpha}\dot{\beta}\dot{\gamma}} = H_{\alpha\beta k}^{i j \dot{\gamma}} = H_i^{\dot{\alpha}\dot{\beta}k} = 0 . \quad (2.24)$$

The Bianchi identities for H can then be solved (see appendix B). The solution is

$$H_{a\alpha\beta}^{ij} = 0 , \quad H_{aij}^{\dot{\alpha}\dot{\beta}} = 0 , \quad H_{a\alpha j}^{i\dot{\alpha}} = 2(\sigma_a)_\alpha{}^{\dot{\alpha}} \mathcal{G}^i_j , \quad (2.25a)$$

$$H_{ab\alpha}^i = \frac{2i}{3}(\sigma_{ab})_\alpha{}^\beta \nabla_\beta^k \mathcal{G}^i_k , \quad H_{abi}^{\dot{\alpha}} = \frac{2i}{3}(\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\nabla}_k^{\dot{\beta}} \mathcal{G}^k_i , \quad (2.25b)$$

$$H_{abc} = \frac{i}{24}\varepsilon_{abcd}(\sigma^d)^\alpha{}_{\dot{\beta}}[\nabla_\alpha^i, \bar{\nabla}_j^{\dot{\beta}}]\mathcal{G}^j_i = \varepsilon_{abcd}\tilde{H}^d , \quad (2.25c)$$

where \mathcal{G}^{ij} is a real symmetric conformally primary dimension-2 superfield, *i.e.*

$$K_A \mathcal{G}^{ij} = 0 , \quad \mathbb{D} \mathcal{G}^{ij} = 2\mathcal{G}^{ij} , \quad Y \mathcal{G}^{ij} = 0 , \quad (\mathcal{G}^{ij})^* = \mathcal{G}_{ij} = \varepsilon_{ik}\varepsilon_{jl}\mathcal{G}^{kl} , \quad (2.26)$$

obeying the constraint

$$\bar{\nabla}_{\dot{\alpha}}^{(i} \mathcal{G}^{jk)} = \nabla_\alpha^{(i} \mathcal{G}^{jk)} = 0 . \quad (2.27)$$

Such a superfield \mathcal{G}^{ij} contains the $\mathcal{N} = 2$ tensor multiplet [42, 43].

It is possible to construct a superfield which automatically obeys the above constraints (2.27) by imposing constraints on the two-form B_{AB} itself [40, 41]. One chooses

$$B_{\alpha\beta}^{ij} = 4i\varepsilon^{ij}\varepsilon_{\alpha\beta}\bar{\Psi} , \quad B_i^{\dot{\alpha}\dot{\beta}} = 4i\varepsilon_{ij}\varepsilon^{\dot{\alpha}\dot{\beta}}\Psi , \quad B_{\alpha j}^{i\dot{\beta}} = 0 , \quad (2.28)$$

where Ψ is a chiral superfield, $\bar{\nabla}_i^{\dot{\alpha}}\Psi = 0$, of dimension 1 and $U(1)_R$ weight 2, but otherwise arbitrary. Such constraints are quite natural since the gauge transformation $\delta B = dV = F$ amounts to

$$\delta\Psi = -\frac{i}{2}\mathcal{W} . \quad (2.29)$$

We can then proceed to solve (2.22) for the full two-form B (see appendix B):

$$B_{a\alpha}^i = (\sigma_a)_\alpha^{\dot{\alpha}} \bar{\nabla}_{\dot{\alpha}}^i \bar{\Psi} , \quad B_{a\dot{i}}^{\dot{\alpha}} = (\sigma_a)_{\dot{\alpha}}^{\alpha} \nabla_i^{\alpha} \Psi , \quad (2.30a)$$

$$B_{ab} = -\frac{i}{4} (\sigma_{ab})^{\alpha\beta} (\nabla_{\alpha\beta} \Psi - 4W_{\alpha\beta} \bar{\Psi}) - \frac{i}{4} (\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}} (\bar{\nabla}^{\dot{\alpha}\dot{\beta}} \bar{\Psi} - 4\bar{W}^{\dot{\alpha}\dot{\beta}} \Psi) . \quad (2.30b)$$

Inserting this solution into the definition of H leads to an expression for the tensor multiplet in terms of an unconstrained chiral prepotential [44–46],

$$\mathcal{G}^{ij} = \frac{1}{4} \nabla^{ij} \Psi + \frac{1}{4} \bar{\nabla}^{ij} \bar{\Psi} . \quad (2.31)$$

One can check that \mathcal{G}^{ij} indeed obeys (2.26) and (2.27) and is invariant under (2.29).

2.4 SU(2) superspace geometry

The conformal superspace geometry presented earlier realizes the superconformal algebra covariantly. Such a formulation proves quite advantageous in finding exact agreement with results derived from $\mathcal{N} = 2$ superconformal tensor calculus. However, one can derive other equally valid formulations via gauge-fixing the additional superspace symmetries [10, 14].

In this section we present the superspace formulation for $\mathcal{N} = 2$ conformal supergravity developed in [10] as a gauge fixed version of the formulation of conformal supergravity in the previous section. In [14], it was described how to do this by the method of degauging, resulting in the superspace geometry with structure group $\text{SL}(2, \mathbb{C}) \times \text{U}(2)_{\text{R}}$; the analysis of [35] could then be applied to reduce to $\text{SU}(2)$ superspace. Here we give an alternative approach which emphasizes how both $\text{U}(2)$ and $\text{SU}(2)$ superspace may be understood as conformal supergravity coupled to a compensator field at the superspace level.

We begin by introducing a real primary superfield X of dimension 2,

$$\mathbb{D}X = 2X , \quad YX = 0 , \quad K^A X = 0 . \quad (2.32)$$

Using this superfield, it is possible to deform the curvatures in a way as to render new covariant derivatives which are completely *inert* under dilatation, conformal, and S -supersymmetry transformations. The new covariant derivatives are given by

$$\mathcal{D}_\alpha^i = e^{-U/4} \left(\nabla_\alpha^i - \nabla^{\beta i} U M_{\beta\alpha} + \frac{1}{4} \nabla_\alpha^i U Y - \nabla_\alpha^j U J_j^i \right) , \quad (2.33a)$$

$$\bar{\mathcal{D}}_{\dot{i}}^{\dot{\alpha}} = e^{-U/4} \left(\bar{\nabla}_{\dot{i}}^{\dot{\alpha}} + \bar{\nabla}_{\dot{\beta}i} U \bar{M}^{\dot{\beta}\dot{\alpha}} - \frac{1}{4} \bar{\nabla}_{\dot{i}}^{\dot{\alpha}} U Y + \bar{\nabla}_{\dot{j}}^{\dot{\alpha}} U J_{\dot{j}}^{\dot{i}} \right) , \quad (2.33b)$$

where $U := \log X$. We refer to these as the X -associated derivatives.⁶ These are constructed so that if Ψ is some conformally primary tensor superfield of vanishing dilatation weight, then $\mathcal{D}_\alpha^i \Psi$ and $\bar{\mathcal{D}}_{\dot{i}}^{\dot{\alpha}} \Psi$ are as well.

⁶The $\mathcal{N} = 1$ analogue of this construction first appeared in [47] and is related to a similar construction in [36]. See [48] for a recent discussion of the analogous construction for $\mathcal{N} = 1$ superspace.

When acting on a conformally primary dimensionless tensor, the algebra of the covariant derivatives becomes

$$\begin{aligned} \{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} &= 4S^{ij}M_{\alpha\beta} + 2\varepsilon^{ij}\varepsilon_{\alpha\beta}Y^{\gamma\delta}M_{\gamma\delta} + 2\varepsilon^{ij}\varepsilon_{\alpha\beta}\bar{W}'_{\dot{\gamma}\dot{\delta}}\bar{M}^{\dot{\gamma}\dot{\delta}} \\ &\quad + 2\varepsilon^{ij}\varepsilon_{\alpha\beta}S^{kl}J_{kl} + 4Y_{\alpha\beta}J^{ij} , \end{aligned} \quad (2.34a)$$

$$\begin{aligned} \{\bar{\mathcal{D}}_i^{\dot{\alpha}}, \bar{\mathcal{D}}_j^{\dot{\beta}}\} &= -4\bar{S}_{ij}\bar{M}^{\dot{\alpha}\dot{\beta}} - 2\varepsilon_{ij}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{Y}_{\dot{\gamma}\dot{\delta}}\bar{M}^{\dot{\gamma}\dot{\delta}} - 2\varepsilon_{ij}\varepsilon^{\dot{\alpha}\dot{\beta}}W'^{\gamma\delta}M_{\gamma\delta} \\ &\quad - 2\varepsilon_{ij}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{S}^{kl}J_{kl} - 4\bar{Y}^{\dot{\alpha}\dot{\beta}}J_{ij} , \end{aligned} \quad (2.34b)$$

where the curvature superfields are defined by

$$S^{ij} := \frac{1}{4X^{3/2}}\nabla^{ij}X , \quad \bar{S}_{ij} := \frac{1}{4X^{3/2}}\bar{\nabla}_{ij}X , \quad (2.35a)$$

$$Y_{\alpha\beta} := -\frac{X^{1/2}}{4}\nabla_{\alpha\beta}X^{-1} , \quad \bar{Y}_{\dot{\alpha}\dot{\beta}} := -\frac{X^{1/2}}{4}\bar{\nabla}_{\dot{\alpha}\dot{\beta}}X^{-1} , \quad (2.35b)$$

$$W'_{\alpha\beta} := X^{-1/2}W_{\alpha\beta} , \quad \bar{W}'_{\dot{\alpha}\dot{\beta}} := X^{-1/2}\bar{W}_{\dot{\alpha}\dot{\beta}} . \quad (2.35c)$$

We have redefined $W_{\alpha\beta}$ and its conjugate to absorb factors of the compensator field and render them inert under dilatations. Henceforth, we will drop the primes. The superfields S^{ij} and $Y_{\alpha\beta}$ appearing above are the only two conformally invariant combinations of two like-chirality spinor derivatives acting on X . For two derivatives of opposite chirality, there are two conformally invariant combinations:

$$G_{\alpha\dot{\alpha}} := -\frac{1}{16}X^{1/2}[\nabla_\alpha^k, \bar{\nabla}_{\dot{\alpha}k}]X^{-1} , \quad G_{\alpha\dot{\alpha}}{}^{ij} := -\frac{i}{8}X^{-1/2}[\nabla_\alpha^{(i}, \bar{\nabla}_{\dot{\alpha}}^{j)}]U . \quad (2.36)$$

Now we construct $\mathcal{D}_{\alpha\dot{\alpha}} = (\sigma^a)_{\alpha\dot{\alpha}}\mathcal{D}_a$. The precise definition of $\mathcal{D}_{\alpha\dot{\alpha}}$ depends, as usual, on conventional constraints. For the usual constraints in $U(2)$ superspace, one takes

$$\begin{aligned} \mathcal{D}_\alpha{}^{\dot{\alpha}} &:= X^{-1/2}\nabla_\alpha{}^{\dot{\alpha}} - \frac{i}{2}X^{-1/4}\nabla_\alpha^k U \bar{\mathcal{D}}_k^{\dot{\alpha}} - \frac{i}{2}X^{-1/4}\bar{\nabla}_k^{\dot{\alpha}} U \mathcal{D}_\alpha^k \\ &\quad - \left(\frac{i}{4}X^{-1/2}\bar{\nabla}_k^{\dot{\alpha}}\nabla^{\beta k}U + 2iG^{\dot{\alpha}\beta} \right) M_{\beta\alpha} + \left(\frac{i}{4}X^{-1/2}\nabla_\alpha^k\bar{\nabla}_{\dot{\beta}k}U - 2iG_{\alpha\dot{\beta}} \right) \bar{M}^{\dot{\beta}\dot{\alpha}} \\ &\quad - i \left(\frac{1}{16}X^{-1/2}[\nabla_\alpha^k, \bar{\nabla}_k^{\dot{\alpha}}]U - G_\alpha{}^{\dot{\alpha}} \right) Y + \frac{i}{2}X^{-1/2}\nabla_\alpha^k U \bar{\nabla}_j^{\dot{\alpha}} U J^j{}_k . \end{aligned} \quad (2.37)$$

This leads to the anti-commutator

$$\begin{aligned} \{\mathcal{D}_\alpha^i, \bar{\mathcal{D}}_j^{\dot{\alpha}}\} &= -2i\delta_j^i\mathcal{D}_\alpha{}^{\dot{\alpha}} - 2(G_\alpha{}^{\dot{\alpha}}\delta_j^i + iG_\alpha{}^{\dot{\alpha}i}{}_j)Y \\ &\quad + 4(G_{\alpha\dot{\beta}}\delta_j^i + iG_{\alpha\dot{\beta}}{}^i{}_j)\bar{M}^{\dot{\beta}\dot{\alpha}} + 4(G^{\dot{\alpha}\beta}\delta_j^i + iG^{\dot{\alpha}\beta i}{}_j)M_{\beta\alpha} \\ &\quad + 8G_\alpha{}^{\dot{\alpha}}J^i{}_j + 4i\delta_j^iG_\alpha{}^{\dot{\alpha}k}{}_l J^l{}_k . \end{aligned} \quad (2.38)$$

To recover $SU(2)$ superspace, we choose $X = \Phi\Phi^\dagger$ where Φ is a chiral superfield of unit dimension. This implies that $G_{\alpha\dot{\alpha}}{}^{ij}$ must vanish. Then modifying slightly the definition of the vector derivative [35]

$$\mathcal{D}'_a = \mathcal{D}_a - iG_a Y , \quad (2.39)$$

leads to a new algebra

$$\{\mathcal{D}_\alpha^i, \bar{\mathcal{D}}_j^{\dot{\alpha}}\} = -2i\delta_j^i \mathcal{D}'_\alpha{}^{\dot{\alpha}} + 4G_{\alpha\dot{\beta}}\delta_j^i \bar{M}^{\dot{\beta}\dot{\alpha}} + 4G^{\dot{\alpha}\beta}\delta_j^i M_{\beta\alpha} + 8G_\alpha{}^{\dot{\alpha}} J^i{}_j, \quad (2.40)$$

without any $U(1)_R$ curvature [10].

Formally, this geometry is not *precisely* that of [10, 34] since there remains a $U(1)_R$ gauge symmetry. However, because there are no $U(1)_R$ curvatures, it is possible to eliminate it completely as discussed in [35]. The way to accomplish that in our language is to use the ratio Φ/Φ^\dagger as a $U(1)_R$ compensator. Applying a similarity transformation to all fields Ψ and operators \mathcal{O} ,

$$\Psi \rightarrow \left(\frac{\Phi}{\Phi^\dagger}\right)^{Y/4} \Psi, \quad \mathcal{O} \rightarrow \left(\frac{\Phi}{\Phi^\dagger}\right)^{Y/4} \mathcal{O} \left(\frac{\Phi}{\Phi^\dagger}\right)^{-Y/4} \quad (2.41)$$

gives a $U(1)_R$ invariant theory. The new derivatives \mathcal{D}_A are given by

$$\mathcal{D}_\alpha^i = (\Phi^\dagger)^{-1/2} \left(\nabla_\alpha^i - (\nabla^{\beta i} \log \Phi) M_{\beta\alpha} - (\nabla_\alpha^j \log \Phi) J_j^i \right), \quad (2.42a)$$

$$\bar{\mathcal{D}}_i^{\dot{\alpha}} = (\Phi)^{-1/2} \left(\bar{\nabla}_i^{\dot{\alpha}} + (\bar{\nabla}_{\dot{\beta} i} \log \Phi^\dagger) \bar{M}^{\dot{\beta}\dot{\alpha}} + (\bar{\nabla}_j^{\dot{\alpha}} \log \Phi^\dagger) J^j{}_i \right), \quad (2.42b)$$

$$\begin{aligned} \mathcal{D}_\alpha{}^{\dot{\alpha}} &:= X^{-1/2} \nabla_\alpha{}^{\dot{\alpha}} - \frac{i}{2} (\Phi^\dagger)^{-1/2} (\nabla_\alpha^k \log \Phi) \bar{\mathcal{D}}_k^{\dot{\alpha}} - \frac{i}{2} \Phi^{-1/2} (\bar{\nabla}_k^{\dot{\alpha}} \log \Phi^\dagger) \mathcal{D}_\alpha^k \\ &\quad - \left(X^{-1/2} \nabla^{\dot{\alpha}\beta} \log \Phi + 2iG^{\dot{\alpha}\beta} \right) M_{\beta\alpha} + \left(X^{-1/2} \nabla_{\alpha\dot{\beta}} \log \Phi^\dagger - 2iG_{\alpha\dot{\beta}} \right) \bar{M}^{\dot{\beta}\dot{\alpha}} \\ &\quad + \frac{i}{2} X^{-1/2} (\nabla_\alpha^k \log \Phi) (\bar{\nabla}_j^{\dot{\alpha}} \log \Phi^\dagger) J^j{}_k, \end{aligned} \quad (2.42c)$$

while the torsion tensors are

$$S^{ij} := \frac{1}{4} \frac{1}{\Phi \Phi^\dagger} \nabla^{ij} \Phi, \quad \bar{S}_{ij} := \frac{1}{4} \frac{1}{\Phi \Phi^\dagger} \bar{\nabla}_{ij} \Phi^\dagger, \quad (2.43a)$$

$$Y_{\alpha\beta} := -\frac{1}{4} \frac{\Phi}{\Phi^\dagger} \nabla_{\alpha\beta} \Phi^{-1}, \quad \bar{Y}_{\dot{\alpha}\dot{\beta}} := -\frac{1}{4} \frac{\Phi^\dagger}{\Phi} \bar{\nabla}_{\dot{\alpha}\dot{\beta}} (\Phi^\dagger)^{-1}, \quad (2.43b)$$

$$W'_{\alpha\beta} := \Phi^{-1} W_{\alpha\beta}, \quad \bar{W}'_{\dot{\alpha}\dot{\beta}} := (\Phi^\dagger)^{-1} \bar{W}_{\dot{\alpha}\dot{\beta}}, \quad (2.43c)$$

$$G_{\alpha\dot{\alpha}} := -\frac{1}{16} X^{1/2} [\nabla_\alpha^k, \bar{\nabla}_{\dot{\alpha}k}] X^{-1}. \quad (2.43d)$$

These obey a number of Bianchi identities [10]:

$$\begin{aligned} \mathcal{D}_\alpha^{(i} S^{jk)} &= \bar{\mathcal{D}}_{\dot{\alpha}}^{(i} S^{jk)} = 0, & \mathcal{D}_i^{(\alpha} Y^{\beta\gamma)} &= 0, & \bar{\mathcal{D}}_i^{\dot{\alpha}} W_{\beta\gamma} &= 0, \\ \mathcal{D}_\alpha^i S_{ij} &= -\mathcal{D}_j^\beta Y_{\beta\alpha}, & \mathcal{D}_\alpha^i G_{\beta\dot{\beta}} &= -\frac{1}{4} \bar{\mathcal{D}}_{\dot{\beta}}^i Y_{\alpha\beta} + \frac{1}{12} \varepsilon_{\alpha\beta} \bar{\mathcal{D}}_{\dot{\beta}j} S^{ji} - \frac{1}{4} \varepsilon_{\alpha\beta} \bar{\mathcal{D}}^{\dot{\gamma}i} \bar{W}_{\dot{\beta}\dot{\gamma}}. \end{aligned} \quad (2.44)$$

A restricted version of $SU(2)$ superspace is found if Φ is further assumed to be a reduced chiral superfield \mathcal{W} (*i.e.* an abelian vector multiplet). In this case S^{ij} is also real [35].

As was shown in [10], $SU(2)$ superspace geometry admits a certain super-Weyl transformation, which means that it may also describe conformal supergravity. In the formulation used here, this super-Weyl transformation corresponds to a reparametrization of the chiral

compensator, $\Phi \rightarrow \Phi e^{-\sigma}$. If Ψ is a conformally primary superfield with dilatation weight Δ and $U(1)_R$ weight w in conformal superspace, then it may be associated with a superfield Ψ' in $SU(2)$ superspace given by

$$\Psi' = \Phi^{-\Delta/2+w/4} (\Phi^\dagger)^{-\Delta/2-w/4} \Psi . \quad (2.45)$$

Ψ' is inert under the dilatation and $U(1)_R$ symmetries of conformal superspace; in their place we have the super-Weyl transformations

$$\Psi' \rightarrow \exp \left(\frac{\Delta}{2} (\sigma + \bar{\sigma}) - \frac{w}{4} (\sigma - \bar{\sigma}) \right) \Psi' . \quad (2.46)$$

Any action which is inert under super-Weyl transformations is necessarily superconformal. At the same time, $SU(2)$ superspace may be used to describe actions which are *not* superconformal.

To avoid unnecessarily long constructions (*e.g.* “the conformal supergravity introduced in [10] based on Grimm’s geometry”) we will refer to any supergeometry with the algebra described in this section as $SU(2)$ superspace, even though, as in the actions considered in [10], the super-Weyl transformation is a symmetry of the action.

3 Component actions from superspace

In the recent paper [14], it was demonstrated that the superconformal geometry in section 2 reduces in components to $\mathcal{N} = 2$ superconformal tensor calculus. This turns out to be one of its main advantages, allowing one to efficiently reproduce the results of tensor calculus computations directly from superspace. Our main goal in this section is to apply these new techniques directly to the computation of a few simple superspace actions involving vector and tensor multiplets.

The component field structure of [14] corresponds to the Weyl multiplet [1–3]. It involves a set of one-forms: the vierbein e_m^a , the gravitino $\psi_{m_i}^\alpha$, the $U(1)_R$ and $SU(2)_R$ gauge fields A_m and ϕ_m^{ij} and the dilatation gauge field b_m , which are each the lowest component of their corresponding superforms:

$$\begin{aligned} e_m^a &:= E_m^a| , & \psi_{m_i}^\alpha &:= 2E_{m_i}^\alpha| , & \bar{\psi}_{m\dot{\alpha}}^i &:= 2E_{m\dot{\alpha}}^i| , \\ A_m &:= \Phi_m| , & \phi_m^{ij} &:= \Phi_m^{ij}| , & b_m &:= B_m| . \end{aligned} \quad (3.1)$$

We denote the component projection of a superfield, $V(z)$ by $V(z)| := V(z)|_{\theta=\bar{\theta}=0}$. There are additional *composite* gauge connections: the spin connection ω_m^{ab} and the special conformal and S -supersymmetry connections \mathfrak{f}_m^a and $\phi_{m_\alpha}^i$,

$$\omega_m^{ab} := \Omega_m^{ab}| , \quad \mathfrak{f}_m^a := \mathfrak{F}_m^a| , \quad \phi_{m_\alpha}^i := 2\mathfrak{F}_{m_\alpha}^i| , \quad \bar{\phi}_{m_i}^{\dot{\alpha}} := 2\mathfrak{F}_{m_i}^{\dot{\alpha}}| , \quad (3.2)$$

which are defined in terms of the other fields. (See appendix C for a summary of these relations.) Finally, in order to have the same number of bosonic and fermionic degrees

of freedom it is necessary to have additional non-gauge fields. These are encoded in the components of the superfield $W_{\alpha\beta}$:

$$W_{ab} = W_{ab}^+ + W_{ab}^-, \quad W_{ab}^+ := (\sigma_{ab})^{\alpha\beta} W_{\alpha\beta}|, \quad W_{ab}^- := -(\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\alpha}\dot{\beta}}|, \quad (3.3a)$$

$$\Sigma^{\alpha i} := \frac{1}{3} \nabla_{\dot{\beta}}^i W^{\alpha\beta}|, \quad \bar{\Sigma}_{\dot{\alpha}i} := -\frac{1}{3} \bar{\nabla}_{\dot{\beta}}^{\dot{\beta}} \bar{W}_{\dot{\alpha}\dot{\beta}}|, \quad (3.3b)$$

$$D := \frac{1}{12} \nabla^{\alpha\beta} W_{\alpha\beta}| = \frac{1}{12} \bar{\nabla}_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\alpha}\dot{\beta}}|, \quad (3.3c)$$

where W_{ab}^{\pm} satisfies the self-duality relation $\frac{i}{2} \varepsilon_{ab}^{cd} W_{cd}^{\pm} = \pm W_{ab}^{\pm}$. All other components of $W_{\alpha\beta}$ can be expressed in terms of the previously defined fields [14]. We will frequently abuse notation and use $W_{\alpha\beta}$ both to denote the superfield and its lowest component. It should be clear from context to which object we are referring.

The situation differs in $SU(2)$ superspace. After having absorbed the compensator, the structure group is effectively reduced to $SL(2, \mathbb{C}) \times SU(2)_R$ with the covariant derivative

$$\mathcal{D}_A = E_A^M \partial_M + \frac{1}{2} \Omega_A^{bc} M_{bc} + \Phi_A^{ij} J_{ij} \quad (3.4)$$

and torsion superfields

$$S^{ij}, \quad Y_{\alpha\beta}, \quad G_{\alpha\dot{\alpha}}, \quad W_{\alpha\beta}, \quad (3.5)$$

together with their complex conjugates. Grimm chose the independent component fields to be [34]

$$\begin{aligned} e_m^a &:= E_m^a|, \quad \psi_{m_i}^\alpha := 2E_m^\alpha|, \quad \bar{\psi}_{m_{\dot{\alpha}}}^{\dot{\alpha}} := 2E_{m_{\dot{\alpha}}}^{\dot{\alpha}}|, \quad \phi_m^{ij} := \Phi_m^{ij}|, \\ W_{\alpha\beta}|, \quad \bar{W}_{\dot{\alpha}\dot{\beta}}|, \quad S^{ij}|, \quad \bar{S}^{ij}|, \quad Y_{\alpha\beta}|, \quad \bar{Y}_{\dot{\alpha}\dot{\beta}}|, \quad G_a|, \\ \rho_{\alpha i} &:= \frac{1}{3} \mathcal{D}_\alpha^j S_{ij}|, \quad \bar{\rho}^{\dot{\alpha}i} := \frac{1}{3} \bar{\mathcal{D}}_{\dot{j}}^{\dot{\alpha}} \bar{S}^{ij}|, \quad \kappa_{\alpha i} := \frac{1}{3} \mathcal{D}_\alpha^j \bar{S}_{ij}|, \quad \bar{\kappa}^{\dot{\alpha}i} := \frac{1}{3} \bar{\mathcal{D}}_{\dot{j}}^{\dot{\alpha}} S^{ij}|, \\ S &:= \mathcal{D}^{ij} \bar{S}_{ij}| + \bar{\mathcal{D}}^{ij} S_{ij}|, \quad C = \frac{1}{12} \mathcal{D}^{ij} S_{ij}|, \quad \bar{C} = \frac{1}{12} \bar{\mathcal{D}}^{ij} \bar{S}_{ij}|. \end{aligned} \quad (3.6)$$

From now on, we will use $W^{\alpha\beta}$, S^{ij} , $Y^{\alpha\beta}$, and G_a to refer to both the superfields and their component projections.

In transforming from the superconformal components to Grimm's components, the $U(1)_R$ connection A_m is replaced by $e_m^a G_a$, the fields Σ_α^i and D are replaced by κ_α^i and S , and the dilatation connection b_m is eliminated by a choice of K_a -gauge. The lowest bosonic and fermionic components of Φ compensate for dilatations, $U(1)_R$ gauge transformations, and S -supersymmetry. The remaining components of this multiplet correspond to S^{ij} , $Y_{\alpha\beta}$, $\rho_{\alpha i}$ and C . If the action under consideration is invariant under super-Weyl transformations, one can show that these component fields must always drop out.

It should be emphasized that Grimm's choice of $\kappa_{\alpha i}$ and S as component fields is purely conventional; it is perhaps more natural to trade these for Σ_α^i and D . The precise

translation is given by

$$\begin{aligned}
\kappa^{\alpha i} &= -\Sigma^{\alpha i} - \frac{1}{3}(\sigma^{ab})^{\alpha\beta}\Psi_{ab\beta}^i + i\psi_a^{\alpha i}G^a - \frac{4i}{3}(\sigma^{ab})_{\beta}^{\alpha}\psi_a^{\beta i}G_b \\
&\quad + \frac{i}{2}(\sigma^a)^{\alpha}_{\dot{\alpha}}\bar{\psi}_{a\dot{j}}^{\dot{\alpha}}\bar{S}^{ij} + \frac{i}{6}(\sigma^a)^{\beta\dot{\alpha}}\bar{\psi}_{a\dot{\alpha}}^i W_{\beta}^{\alpha} + \frac{i}{2}(\sigma^a)^{\alpha}_{\dot{\beta}}\bar{\psi}_{a\dot{\alpha}}^i \bar{Y}^{\dot{\alpha}\dot{\beta}}, \\
S &= 12D + 12Y^{\alpha\beta}W_{\alpha\beta} - 12S^{ij}\bar{S}_{ij} - 4\mathcal{R} - 48G^a G_a \\
&\quad - 4i(\psi^b_k \sigma^a \bar{\Psi}_{ab}^k) + 8(\psi^b_k \sigma^a \bar{\psi}_b^k)G_a + 4(\psi_{ak} \sigma^a \bar{\psi}_b^k)G^b \\
&\quad - 6\psi_{ai}\psi^a_j S^{ij} - 2(\sigma^a)_{(\alpha}(\sigma^b)_{\beta)}^{\dot{\beta}}\psi_{a\dot{\alpha}}^{\alpha}\psi_{b\dot{\beta}}^{\beta k}\bar{W}_{\dot{\alpha}\dot{\beta}} + 6\psi_{ak}^{\gamma}\psi^{a\delta k}Y_{\gamma\delta} + \text{c.c.}, \quad (3.7)
\end{aligned}$$

where $\mathcal{R} = \mathcal{R}_{ab}{}^{ab}$ is the Ricci scalar and Ψ_{abk}^{γ} is the gravitino field strength, defined in appendix C.

3.1 General component chiral actions from superspace

The conformal supergravity formulations presented in section 2 allow the construction of superconformally invariant actions [14]. The simplest invariants are integrals over the full superspace

$$S = \int d^{12}z E \mathcal{L}, \quad d^{12}z \equiv d^4x d^4\theta d^4\bar{\theta}, \quad (3.8)$$

where $E := \text{sdet}(E_M^A)$ and \mathcal{L} is a real conformally primary $U(2)_R$ scalar with vanishing conformal dimension,

$$\mathbb{D}\mathcal{L} = Y\mathcal{L} = J^{ij}\mathcal{L} = M_{ab}\mathcal{L} = K_a\mathcal{L} = S_{\alpha}^i\mathcal{L} = \bar{S}_{\dot{i}}^{\dot{\alpha}}\mathcal{L} = 0. \quad (3.9)$$

However, these tend to yield higher derivative actions and play less of a role in $\mathcal{N} = 2$ superspace than in $\mathcal{N} = 1$. Of more importance is the chiral action, which involves an integral over the chiral subspace

$$S = S_c + \text{c.c.}, \quad S_c = \int d^8z \mathcal{E} \mathcal{L}_c, \quad d^8z := d^4x d^4\theta, \quad (3.10)$$

where \mathcal{L}_c is chiral, $\bar{\nabla}_{\dot{i}}^{\dot{\alpha}}\mathcal{L}_c = 0$ and \mathcal{E} is a suitably chosen chiral measure [14]. The Lagrangian \mathcal{L}_c must be a conformally primary Lorentz and $SU(2)_R$ chiral scalar with conformal dimension two and $U(1)_R$ weight -4 :

$$\mathbb{D}\mathcal{L}_c = 2\mathcal{L}_c, \quad Y\mathcal{L}_c = -4\mathcal{L}_c, \quad J^{ij}\mathcal{L}_c = M_{ab}\mathcal{L}_c = K_a\mathcal{L}_c = S_{\alpha}^i\mathcal{L}_c = \bar{S}_{\dot{i}}^{\dot{\alpha}}\mathcal{L}_c = 0. \quad (3.11)$$

Any action involving an integral over the full superspace may be converted to one over the chiral subspace by the rule [14]

$$\int d^{12}z E \mathcal{L} = \int d^8z \mathcal{E} \bar{\nabla}^4 \mathcal{L}, \quad \bar{\nabla}^4 := \frac{1}{48} \bar{\nabla}^{ij} \bar{\nabla}_{ij}. \quad (3.12)$$

In the rest of this paper we will consider only chiral actions when dealing with conventional $\mathcal{N} = 2$ superspace.

It was shown in [14] that the component action for a chiral action is

$$\begin{aligned}
S_c = \int d^4x e \left(\frac{1}{48} \nabla^{ij} \nabla_{ij} - \frac{i}{12} \bar{\psi}_{d\dot{\delta}}^l (\tilde{\sigma}^d)^{\dot{\delta}\alpha} \nabla_\alpha^q \nabla_{lq} + \frac{i}{2} \bar{\psi}_{d\dot{\delta}}^l (\sigma^d)_{\alpha\dot{\alpha}} \bar{W}^{\dot{\alpha}\dot{\delta}} \nabla_l^\alpha + \bar{W}^{\dot{\alpha}\dot{\beta}} \bar{W}_{\dot{\alpha}\dot{\beta}} \right. \\
+ \frac{1}{4} \bar{\psi}_{c\dot{\gamma}}^k \bar{\psi}_{d\dot{\delta}}^l \left((\tilde{\sigma}^{cd})^{\dot{\gamma}\dot{\delta}} \nabla_{kl} - \frac{1}{2} \varepsilon^{\dot{\gamma}\dot{\delta}} \varepsilon_{kl} (\sigma^{cd})_{\beta\gamma} \nabla^{\beta\gamma} - 4 \varepsilon^{\dot{\gamma}\dot{\delta}} \varepsilon_{kl} (\tilde{\sigma}^{cd})_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\alpha}\dot{\beta}} \right) \\
\left. - \frac{1}{4} \varepsilon^{abcd} (\tilde{\sigma}_a)^{\dot{\beta}\alpha} \bar{\psi}_{b\dot{\beta}}^j \bar{\psi}_{c\dot{\gamma}}^k \bar{\psi}_{d\dot{\delta}}^l \nabla_{\alpha j} - \frac{i}{4} \varepsilon^{abcd} \bar{\psi}_{a\dot{\alpha}}^i \bar{\psi}_{b\dot{\beta}}^{\dot{\alpha}} \bar{\psi}_{c\dot{\gamma}}^j \bar{\psi}_{d\dot{\delta}}^{\dot{\beta}} \right) \mathcal{L}_c \Big| . \quad (3.13)
\end{aligned}$$

This corresponds to the action of a chiral multiplet coupled to $\mathcal{N} = 2$ conformal supergravity constructed in [49].

When dealing with actions which are not superconformal, one must resort to a different superspace geometry, or equivalently, introduce compensators. For example, in $SU(2)$ superspace, one may introduce the chiral action⁷

$$S_c = \int d^4x d^4\theta \mathcal{E} \mathcal{L}_c . \quad (3.14)$$

Its component form is given by

$$\begin{aligned}
S_c = \int d^4x e \left[\frac{1}{48} \mathcal{D}^{ij} \mathcal{D}_{ij} + \frac{7}{12} S^{ij} \mathcal{D}_{ij} - \frac{1}{4} Y^{\alpha\beta} \mathcal{D}_{\alpha\beta} + \frac{1}{2} (\mathcal{D}_i^\alpha S^{ij}) \mathcal{D}_{\alpha j} + \frac{1}{6} (\mathcal{D}^{ij} S_{ij}) \right. \\
+ 3 S^{ij} S_{ij} - Y^{\alpha\beta} Y_{\alpha\beta} + \bar{W}^{\dot{\alpha}\dot{\beta}} \bar{W}_{\dot{\alpha}\dot{\beta}} \\
- \frac{i}{6} \bar{\psi}_{d\dot{\delta}}^l (\tilde{\sigma}^d)^{\dot{\delta}\alpha} \left(\frac{1}{2} \mathcal{D}_\alpha^q \mathcal{D}_{lq} + 5 S_{lq} \mathcal{D}_\alpha^q - 3 Y_{\alpha\beta} \mathcal{D}_l^\beta + 4 (\mathcal{D}_\alpha^q S_{lq}) \right) + \frac{i}{2} \bar{\psi}_{d\dot{\delta}}^l (\sigma^d)_{\alpha\dot{\alpha}} \bar{W}^{\dot{\alpha}\dot{\delta}} \mathcal{D}_l^\alpha \\
+ \frac{1}{4} \bar{\psi}_{c\dot{\gamma}}^k \bar{\psi}_{d\dot{\delta}}^l \left((\tilde{\sigma}^{cd})^{\dot{\gamma}\dot{\delta}} \mathcal{D}_{kl} + 8 (\tilde{\sigma}^{cd})^{\dot{\gamma}\dot{\delta}} S_{kl} - 4 \varepsilon^{\dot{\gamma}\dot{\delta}} \varepsilon_{kl} (\tilde{\sigma}^{cd})_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\alpha}\dot{\beta}} \right. \\
\left. - \frac{1}{2} \varepsilon^{\dot{\gamma}\dot{\delta}} \varepsilon_{kl} (\sigma^{cd})_{\beta\gamma} \mathcal{D}^{\beta\gamma} - 4 \varepsilon^{\dot{\gamma}\dot{\delta}} \varepsilon_{kl} (\sigma^{cd})_{\alpha\beta} Y^{\alpha\beta} \right) \\
\left. - \frac{1}{4} \varepsilon^{abcd} (\tilde{\sigma}_a)^{\dot{\beta}\alpha} \bar{\psi}_{b\dot{\beta}}^j \bar{\psi}_{c\dot{\gamma}}^k \bar{\psi}_{d\dot{\delta}}^l \mathcal{D}_{\alpha j} - \frac{i}{4} \varepsilon^{abcd} \bar{\psi}_{a\dot{\alpha}}^i \bar{\psi}_{b\dot{\beta}}^{\dot{\alpha}} \bar{\psi}_{c\dot{\gamma}}^j \bar{\psi}_{d\dot{\delta}}^{\dot{\beta}} \right] \mathcal{L}_c \Big| . \quad (3.15)
\end{aligned}$$

If \mathcal{L}_c transforms under super-Weyl transformations as (2.46) with $\Delta = 2$ and $w = -4$, then one can show that the action is super-Weyl invariant. For this case, this action and (3.13) can be shown to coincide.

It is worth noting that the component action (3.15) has been obtained via the superform method [50]; one expects the same should be true of the superconformal component action (3.13).

In the rest of this section we will present component results and actions for the $\mathcal{N} = 2$ abelian vector and tensor multiplets.

⁷It should be mentioned that \mathcal{E} here is defined using the vierbein associated with \mathcal{D}_A .

3.2 The abelian vector multiplet in components

As discussed in subsection 2.2, the abelian vector multiplet is described by a primary reduced chiral superfield \mathcal{W} of dimension 1,

$$\bar{\nabla}_i^{\dot{\alpha}} \mathcal{W} = 0, \quad \nabla^{ij} \mathcal{W} = \bar{\nabla}^{ij} \bar{\mathcal{W}}, \quad \mathbb{D} \mathcal{W} = \mathcal{W}, \quad Y \mathcal{W} = -2 \mathcal{W}, \quad K_A \mathcal{W} = 0. \quad (3.16)$$

The vector field strength F in superspace is given by (2.15) and (2.17). Within the superfield \mathcal{W} are found the matter components of the abelian vector multiplet: a complex scalar field ϕ , a gaugino λ_α^i and a real SU(2) isotriplet X^{ij} ,

$$\phi = \mathcal{W}|, \quad \lambda_\alpha^i = \nabla_\alpha^i \mathcal{W}|, \quad X^{ij} = \nabla^{ij} \mathcal{W}|. \quad (3.17)$$

The reality of X^{ij} follows from the Bianchi identity. The remaining component field, the gauge connection v_m is given by the lowest component of the corresponding superspace connection, $v_m = V_m|$.

The component two-form field strength is constructed from a projection of the superspace two-form,

$$f_{mn} = F_{mn}| = 2\partial_{[m} V_{n]}| = 2\partial_{[m} v_{n]}. \quad (3.18)$$

Making use of the identity

$$F_{mn} = E_m^A E_n^B F_{AB}(-)^{ab} \quad (3.19)$$

and projecting to lowest component, we may solve for $F_{ab}|$ to give

$$\begin{aligned} \hat{F}_{ab} := F_{ab}| = & e_a^m e_b^n f_{mn} - \frac{i}{2} (\sigma_{[a})_\alpha^{\dot{\alpha}} \psi_{b]}^{\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}^k + \frac{i}{2} (\tilde{\sigma}_{[a})_{\dot{\alpha}}^\alpha \lambda_{\alpha}^k \bar{\psi}_{b]}^{\dot{\alpha}} \\ & - \frac{1}{2} \psi_a^\gamma \psi_b^k \bar{\phi} + \frac{1}{2} \bar{\psi}_a^{\dot{\gamma}} \bar{\psi}_b^{\dot{k}} \phi. \end{aligned} \quad (3.20)$$

In the language of the superconformal tensor calculus, \hat{F}_{ab} is referred to as the supercovariant field strength whereas f_{mn} is the conventional field strength. Recall that F_{ab} is given by equation (2.17b).

The supersymmetry transformations for this (or any) multiplet may be derived via a covariant Lie derivative in superspace. For any covariant superfield Ψ , this reduces to the usual covariant derivative:

$$\delta \Psi = \xi_i^\alpha \nabla_\alpha^i \Psi + \bar{\xi}_{\dot{\alpha}}^i \bar{\nabla}_i^{\dot{\alpha}} \Psi. \quad (3.21)$$

Projecting to lowest components yields the transformation of the field $\Psi|$. Applying this rule to $\Psi = \mathcal{W}$ and $\Psi = \nabla_\alpha^i \mathcal{W}$ leads to

$$\delta \phi = \xi_i^\alpha \lambda_\alpha^i \quad (3.22)$$

and

$$\delta \lambda_\alpha^i = -\frac{1}{2} \xi_{\alpha j} X^{ij} - 2(\sigma^{ab})_{\alpha\beta} \xi^{\beta i} \hat{F}_{ab} - 2\xi^{\beta i} W_{\alpha\beta} \bar{\phi} - 2i \bar{\xi}_{\dot{\beta}}^i \nabla_\alpha^{\dot{\beta}} \phi + i \bar{\xi}_{\dot{\beta}}^i \psi_\alpha^{\dot{\beta}} \lambda_\gamma^k, \quad (3.23)$$

where ∇'_a is defined in (3.32). The other supersymmetry transformations may be derived similarly.

The vector multiplet in $SU(2)$ superspace can be easily worked out. We find the same result as in [10]. The constraints on \mathcal{W} are

$$\bar{\mathcal{D}}_i^{\dot{\alpha}} \mathcal{W} = 0, \quad (\mathcal{D}^{ij} + 4S^{ij})\mathcal{W} = (\bar{\mathcal{D}}^{ij} + 4S^{ij})\bar{\mathcal{W}}, \quad (3.24)$$

and the matter fields are

$$\phi = \mathcal{W}|, \quad \lambda_\alpha^i = \mathcal{D}_\alpha^i \mathcal{W}|, \quad X^{ij} = (\mathcal{D}^{ij} + 4S^{ij})\mathcal{W}|. \quad (3.25)$$

One defines the superspace two-form using the derivatives \mathcal{D}_A . The result is formally identical to (2.15) and (2.17), but with the replacements

$$\nabla_\alpha^i \mathcal{W} \rightarrow \mathcal{D}_\alpha^i \mathcal{W}, \quad \nabla_{\alpha\beta} \mathcal{W} \rightarrow \mathcal{D}_{\alpha\beta} \mathcal{W} + 4Y_{\alpha\beta} \mathcal{W}. \quad (3.26)$$

The component two-form is still given by (3.20).

3.3 Abelian vector multiplet action

Let us now consider the general chiral action $\mathcal{L}_c = \mathcal{F}(\mathcal{W}^I)$ of n abelian vector multiplets⁸ \mathcal{W}^I . We will consider its evaluation in two distinct cases, first for the superconformal case and then for the general case.

3.3.1 The superconformal case

In the superconformal case, \mathcal{F} must be homogeneous of degree two in \mathcal{W}^I ,

$$\mathcal{W}^I \mathcal{F}_I \equiv \mathcal{W}^I \frac{\partial}{\partial \mathcal{W}^I} \mathcal{F} = 2\mathcal{F}. \quad (3.27)$$

The component action coupled to conformal supergravity was given first in [51]. Here we describe how to reconstruct that result directly from superspace.

Using the component reduction formula (3.13), it is straightforward to project to components. The following two identities prove useful:

$$\nabla_\alpha^i \nabla_{ij} \mathcal{W}^I = -6i \nabla_{\alpha\dot{\beta}} \bar{\nabla}_j^{\dot{\beta}} \mathcal{W}^I = -6i \bar{\nabla}_j^{\dot{\beta}} \nabla_{\alpha\dot{\beta}} \mathcal{W}^I, \quad (3.28a)$$

$$\nabla^{ij} \nabla_{ij} \mathcal{W}^I = 48\Box \bar{\mathcal{W}}^I + 12\bar{W}_{\dot{\alpha}\dot{\beta}} \bar{\nabla}^{\dot{\beta}\dot{\alpha}} \bar{\mathcal{W}}^I + 12\bar{\nabla}_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\alpha}\dot{\beta}} \bar{\mathcal{W}}^I + 24\bar{\nabla}_{\dot{\beta}j} \bar{W}^{\dot{\beta}\dot{\alpha}} \bar{\nabla}_{\dot{\alpha}}^j \bar{\mathcal{W}}^I, \quad (3.28b)$$

where $\Box = \nabla^a \nabla_a$ is the supercovariant d'Alembertian. When taken to lowest components they yield

$$\nabla_\alpha^i \nabla_{ij} \mathcal{W}^I| = -6i \nabla_{\alpha\dot{\beta}} \bar{\lambda}_j^{I\dot{\beta}}, \quad (3.29a)$$

$$\frac{1}{48} \nabla^{ij} \nabla_{ij} \mathcal{W}^I| = \Box \bar{\phi}^I + 2\bar{W}_{\dot{\alpha}\dot{\beta}} \hat{F}^{\dot{\beta}\dot{\alpha}} - \bar{W}_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\alpha}\dot{\beta}} \phi^I + 3D\bar{\phi}^I + \frac{3}{2} \bar{\Sigma}_j^{\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}^j. \quad (3.29b)$$

⁸The non-abelian case is a straightforward generalization.

The full component action is:

$$\begin{aligned}
S_c = \int d^4x e \Big(& \mathcal{F}_I \square \bar{\phi}^I - \frac{i}{2} \mathcal{F}_{IJ} \lambda^{I\alpha j} \nabla_{\alpha\dot{\alpha}} \bar{\lambda}^{J\dot{\alpha}}_j + \frac{1}{32} \mathcal{F}_{IJ} X^{Iij} X^J_{ij} - 2 \mathcal{F}_{IJ} \hat{F}^{I\alpha\beta} \hat{F}^J_{\alpha\beta} \\
& - \mathcal{F} \bar{W}_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\alpha}\dot{\beta}} - 2 \mathcal{F}_I \bar{W}_{\dot{\alpha}\dot{\beta}} \hat{F}^{I\dot{\alpha}\dot{\beta}} + 3 \mathcal{F}_I \bar{\phi}^I D + \frac{3}{2} \mathcal{F}_I \bar{\Sigma}^k \bar{\lambda}^I_k - 2 \mathcal{F}_{IJ} \bar{\phi}^I W^{\alpha\beta} \hat{F}^J_{\alpha\beta} \\
& - \frac{1}{2} \mathcal{F}_{IJ} \bar{\phi}^I \bar{\phi}^J W^{\alpha\beta} W_{\alpha\beta} + \frac{1}{16} \mathcal{F}_{IJK} (\lambda^{Ii} \lambda^{Jj}) X^K_{ij} + \frac{1}{2} \mathcal{F}_{IJK} \lambda^{I\alpha k} \lambda^{J\beta}_k \hat{F}^K_{\alpha\beta} \\
& + \frac{1}{4} \mathcal{F}_{IJK} \bar{\phi}^I \lambda^{J\alpha k} \lambda^{K\beta}_k W_{\alpha\beta} + \frac{1}{48} \mathcal{F}_{IJKL} (\lambda^{Ii} \lambda^{Jj}) (\lambda^K_i \lambda^L_j) - \frac{1}{2} \mathcal{F}_I (\bar{\psi}_m^j \tilde{\sigma}^m \sigma^b)_{\dot{\alpha}} \nabla_b \bar{\lambda}^{I\dot{\alpha}}_j \\
& + \frac{i}{2} \mathcal{F}_I \bar{\psi}_{b\dot{\gamma}}^j \bar{W}^{\dot{\gamma}}_{\dot{\beta}} \tilde{\sigma}^{\dot{\beta}\alpha}_b \lambda^I_{\alpha j} - \frac{i}{8} \mathcal{F}_{IJ} (\bar{\psi}_m^i \tilde{\sigma}^m \lambda^{Ij}) X^J_{ij} - i \mathcal{F}_{IJ} (\bar{\psi}_m^k \tilde{\sigma}^m)_{\alpha} \lambda^{I\beta}_k \hat{F}^J_{\alpha\beta} \\
& - \frac{i}{2} \mathcal{F}_{IJ} \bar{\phi}^I (\bar{\psi}_m^k \tilde{\sigma}^m)_{\alpha} \lambda^{J\beta}_k W_{\alpha\beta} - \frac{i}{12} \mathcal{F}_{IJK} (\bar{\psi}_m^i \tilde{\sigma}^m \lambda^{Ij}) (\lambda^K_j \lambda^L_i) - \frac{1}{4} \mathcal{F}_I (\bar{\psi}_m^i \tilde{\sigma}^{mn} \bar{\psi}_n^j) X^I_{ij} \\
& + \mathcal{F}_I (\bar{\psi}_m \bar{\psi}_n) (\sigma^{mn})^{\alpha\beta} \hat{F}^I_{\alpha\beta} + \frac{1}{2} \mathcal{F}_I \bar{\phi}^I (\bar{\psi}_m \bar{\psi}_n) (\sigma^{mn})^{\alpha\beta} W_{\alpha\beta} - \mathcal{F} (\bar{\psi}_m \bar{\psi}_n) (\tilde{\sigma}^{mn})^{\dot{\alpha}\dot{\beta}} \bar{W}_{\dot{\alpha}\dot{\beta}} \\
& - \frac{1}{4} \mathcal{F}_{IJ} (\bar{\psi}_m^i \tilde{\sigma}^{mn} \bar{\psi}_n^j) (\lambda^I_i \lambda^J_j) + \frac{1}{8} \mathcal{F}_{IJ} (\bar{\psi}_m \bar{\psi}_n) (\lambda^{Ik} \sigma^{mn} \lambda^J_k) \\
& + \frac{1}{4} \mathcal{F}_I \varepsilon^{mnpq} (\bar{\psi}_m \bar{\psi}_n) (\bar{\psi}_p^i \tilde{\sigma}_q \lambda^I_i) - \frac{i}{4} \mathcal{F} \varepsilon^{mnpq} (\bar{\psi}_m \bar{\psi}_n) (\bar{\psi}_p \bar{\psi}_q) \Big) , \tag{3.30}
\end{aligned}$$

where

$$\nabla_a \phi^I \equiv \nabla'_a \phi^I - \frac{1}{2} (\psi_{aj} \lambda^{Ij}) , \tag{3.31a}$$

$$\nabla_b \bar{\lambda}^{I\dot{\alpha}}_j \equiv \nabla'_b \bar{\lambda}^{I\dot{\alpha}}_j + 2 \bar{\phi}^I \bar{\phi}_{b\dot{j}}^{\dot{\alpha}} - i \psi_{b\beta j} \nabla^{\dot{\alpha}\beta} \bar{\phi}^I + \frac{1}{4} \bar{\psi}_b^{\dot{\alpha}k} X^I_{kj} + 2 \bar{\psi}_{b\dot{\beta}j} \hat{F}^{I\dot{\beta}\dot{\alpha}} + \bar{\psi}_{b\dot{\beta}j} \bar{W}^{\dot{\beta}\dot{\alpha}} \phi^I , \tag{3.31b}$$

$$\begin{aligned}
\square \bar{\phi}^I & \equiv \nabla'^a \nabla_a \bar{\phi}^I + 2 \mathfrak{f}_a^{a} \bar{\phi}^I - \frac{i}{2} (\phi_m^j \sigma^m \bar{\lambda}^I_j) \\
& + \frac{i}{4} (\psi_{mj} \sigma^m)_{\dot{\alpha}} \bar{W}^{\dot{\alpha}\dot{\beta}} \bar{\lambda}^{Ij}_{\dot{\beta}} - \frac{3i}{4} (\psi_{mj} \sigma^m \bar{\Sigma}^j) \bar{\phi}^I - \frac{1}{2} \bar{\psi}_{a\dot{\beta}}^j \nabla^a \bar{\lambda}^{I\dot{\beta}}_j . \tag{3.31c}
\end{aligned}$$

The derivative ∇'_a which we have introduced above is given by

$$\nabla'_a := e_a^{m} \left(\partial_m + \frac{1}{2} \omega_m^{ab} M_{ab} + \phi_m^{ij} J_{ij} + i A_m Y + b_m \mathbb{D} \right) . \tag{3.32}$$

This result may be compared with that given in [51].

3.3.2 The general case

Next, we consider the situation in $SU(2)$ superspace for the action

$$S = \int d^4x d^4\theta \mathcal{E} \mathcal{F}(\mathcal{W}^I) . \tag{3.33}$$

Here \mathcal{F} may be an *arbitrary* function of \mathcal{W}^I , not necessarily of degree 2. The component reduction is straightforward by making use of (3.15) and the results in appendix C. A

number of identities for derivatives of \mathcal{W} are useful:

$$\mathcal{D}^{ij}\mathcal{W} = X^{ij} - 4S^{ij}\mathcal{W} , \quad (3.34a)$$

$$\begin{aligned} \mathcal{D}^{\alpha j}\mathcal{D}_{ij}\mathcal{W} &= 4(\mathcal{D}^{\alpha j}\bar{S}_{ij})\bar{\mathcal{W}} - 4(\mathcal{D}^{\alpha j}S_{ij})\mathcal{W} - 6G^{\alpha\dot{\alpha}}\bar{\mathcal{D}}_{\dot{\alpha}i}\bar{\mathcal{W}} \\ &\quad - 4S_{ij}\mathcal{D}^{\alpha j}\mathcal{W} + 6i\mathcal{D}^{\alpha\dot{\alpha}}\bar{\mathcal{D}}_{\dot{\alpha}i}\bar{\mathcal{W}} , \end{aligned} \quad (3.34b)$$

$$\begin{aligned} \mathcal{D}^{ij}\mathcal{D}_{ij}\mathcal{W} &= 48\mathcal{D}^a\mathcal{D}_a\bar{\mathcal{W}} + 192iG_a\mathcal{D}^a\bar{\mathcal{W}} - 16S^{ij}\bar{\mathcal{D}}_{ij}\bar{\mathcal{W}} + 12\bar{\mathcal{W}}_{\dot{\alpha}\dot{\beta}}\bar{\mathcal{D}}^{\dot{\alpha}\dot{\beta}}\bar{\mathcal{W}} \\ &\quad + 8(\bar{\mathcal{D}}^{\dot{\alpha}i}S_{ij})\bar{\mathcal{D}}_{\dot{\alpha}}^j\bar{\mathcal{W}} - 8\mathcal{D}_i^\alpha S^{ij}\mathcal{D}_{\alpha j}\mathcal{W} - 24(\bar{\mathcal{D}}^{\dot{\alpha}i}\bar{\mathcal{W}}_{\dot{\alpha}\dot{\beta}})\bar{\mathcal{D}}_{\dot{\beta}}^i\bar{\mathcal{W}} \\ &\quad + 4\mathcal{D}^{ij}\bar{S}_{ij}\bar{\mathcal{W}} - 4\mathcal{D}^{ij}S_{ij}\mathcal{W} - 16\bar{S}^{ij}S_{ij}\bar{\mathcal{W}} + 16S^{ij}S_{ij}\mathcal{W} . \end{aligned} \quad (3.34c)$$

In writing the action, and in particular showing that it must be real (up to a total derivative) for the case $\mathcal{F}(\mathcal{W}) = \mathcal{W}^2$, it is helpful to introduce vector derivatives which do not contain the gravitino. As a first step, we introduce

$$\mathcal{D}'_a = e_a{}^m \left(\partial_m + \frac{1}{2}\omega_m{}^{bc}M_{bc} + \phi_m{}^{ij}J_{ij} \right) . \quad (3.35)$$

Note that $\mathcal{D}'_a = e_a{}^m \mathcal{D}_m|$, where $\mathcal{D}_m = E_m{}^A \mathcal{D}_A$ is defined in superspace. We may calculate

$$[\mathcal{D}'_a, \mathcal{D}'_b] = \mathcal{T}_{ab}{}^c(x)\mathcal{D}'_c + \frac{1}{2}\mathcal{R}_{ab}{}^{cd}(x)M_{cd} + \mathcal{R}_{ab}{}^{ij}J_{ij} . \quad (3.36)$$

The torsion tensor $\mathcal{T}_{ab}{}^c$ and the curvature tensors $\mathcal{R}_{ab}{}^{cd}$ and $\mathcal{R}_{ab}{}^{ij}$ coincide with certain projections of superspace torsions and curvatures,

$$\mathcal{T}_{ab}{}^c = e_a{}^m e_b{}^n T_{mn}{}^c| , \quad \mathcal{R}_{ab}{}^{cd} = e_a{}^m e_b{}^n R_{mn}{}^{cd}| , \quad \mathcal{R}_{ab}{}^{ij} = e_a{}^m e_b{}^n R_{mn}{}^{ij}| . \quad (3.37)$$

An important property of these derivatives is that the torsion tensor does not vanish, but is given by a gravitino bilinear, due to the presence of gravitinos in the spin connection. To extract this bilinear, we can introduce the torsion-free covariant derivatives (see *e.g.* [36])

$$\tilde{\mathcal{D}}_a := e_a{}^m \left(\partial_m + \frac{1}{2}\tilde{\omega}_m{}^{bc}M_{bc} + \phi_m{}^{ij}J_{ij} \right) , \quad (3.38)$$

$$\tilde{\omega}_{mbc} := \omega_{mbc} + \frac{1}{2}e_m{}^a(\mathcal{T}_{acb} + \mathcal{T}_{bcd} - \mathcal{T}_{abc}) . \quad (3.39)$$

The new spin connection $\tilde{\omega}_{mbc}$ depends only on the vierbein, $\tilde{\omega}_{mbc} = \tilde{\omega}_{mbc}(e)$, so the algebra of these covariant derivatives is torsion-free

$$[\tilde{\mathcal{D}}_a, \tilde{\mathcal{D}}_b] = \frac{1}{2}\tilde{R}_{ab}{}^{cd}M_{cd} + \mathcal{R}_{ab}{}^{ij}J_{ij} , \quad (3.40)$$

where $\tilde{R}_{ab}{}^{cd}$ depends only on the vierbein, $\tilde{R}_{ab}{}^{cd} = \tilde{R}_{ab}{}^{cd}(e)$.

Now we present the action, separating it in terms of the number of gravitinos appearing explicitly within:

$$S_c = \int d^4x e (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4) , \quad (3.41)$$

where \mathcal{L}_n contains n explicit gravitinos. The terms are

$$\begin{aligned}
\mathcal{L}_0 = & \mathcal{F} \left(2C + 3S^{ij}S_{ij} - Y^{\alpha\beta}Y_{\alpha\beta} + \bar{W}^{\dot{\alpha}\dot{\beta}}\bar{W}_{\dot{\alpha}\dot{\beta}} \right) \\
& + \mathcal{F}_I \left(\frac{1}{24}S\bar{\phi}^I + 2Y^{\alpha\beta}\hat{F}_{\alpha\beta}^I - 2\bar{W}^{\dot{\alpha}\dot{\beta}}\hat{F}_{\dot{\alpha}\dot{\beta}}^I + \frac{1}{4}S^{ij}X_{ij}^I + \tilde{\mathcal{D}}^a\tilde{\mathcal{D}}_a\bar{\phi}^I \right. \\
& \quad + 2i\tilde{\mathcal{D}}_a(G^a\bar{\phi}^I) + 2iG^a\tilde{\mathcal{D}}_a\bar{\phi}^I - C\phi^I - 2S^{ij}S_{ij}\phi^I - \bar{W}^{\dot{\alpha}\dot{\beta}}\bar{W}_{\dot{\alpha}\dot{\beta}}\phi^I \\
& \quad + Y^{\alpha\beta}Y_{\alpha\beta}\phi^I + \frac{1}{2}Y^{\alpha\beta}W_{\alpha\beta}\bar{\phi}^I - \frac{1}{2}\bar{Y}^{\dot{\alpha}\dot{\beta}}\bar{W}_{\dot{\alpha}\dot{\beta}}\bar{\phi}^I + \bar{S}^{ij}S_{ij}\bar{\phi}^I + \rho^i\lambda_i^I + \bar{\kappa}^i\bar{\lambda}_i^I \Big) \\
& + \mathcal{F}_{IJ} \left(\frac{1}{32}X^{Iij}X_{ij}^J - \frac{1}{2}\hat{F}^{Iab}\hat{F}_{ab}^J - \frac{i}{4}\varepsilon^{mnpq}f_{mn}f_{pq} - \frac{1}{4}X^{Iij}S_{ij}\phi^J - 2Y_{\alpha\beta}\hat{F}^{I\alpha\beta}\phi^J \right. \\
& \quad - 2W_{\alpha\beta}\hat{F}^{I\alpha\beta}\bar{\phi}^J + \frac{1}{2}S^{ij}S_{ij}\phi^I\phi^J - \frac{1}{2}Y^{\alpha\beta}Y_{\alpha\beta}\phi^I\phi^J - Y^{\alpha\beta}W_{\alpha\beta}\phi^I\bar{\phi}^J \\
& \quad - \frac{1}{2}W^{\alpha\beta}W_{\alpha\beta}\bar{\phi}^I\bar{\phi}^J + \kappa^i\lambda_i^I\bar{\phi}^J - \rho^i\lambda_i^I\phi^J - \frac{i}{2}\lambda^{I\alpha i}\tilde{\mathcal{D}}_{\alpha\dot{\alpha}}\bar{\lambda}^{J\dot{\alpha}}_i + \frac{1}{2}G^a(\lambda^{Ii}\sigma_a\bar{\lambda}_i^J) \Big) \\
& + \mathcal{F}_{IJK} \left(\frac{1}{16}X_{ij}^I\lambda^{Ji}\lambda^{Kj} + \frac{1}{2}\hat{F}^{I\alpha\beta}\lambda_{\alpha}^{Ji}\lambda_{\beta}^{Kj} - \frac{1}{4}S_{ij}\phi^I\lambda^{Ji}\lambda^{Kj} \right. \\
& \quad \left. + \frac{1}{4}Y^{\alpha\beta}\phi^I\lambda_{\alpha}^{Ji}\lambda_{\beta}^{Kj} + \frac{1}{4}W^{\alpha\beta}\bar{\phi}^I\lambda_{\alpha}^{Ji}\lambda_{\beta}^{Kj} \right) \\
& + \frac{1}{48}\mathcal{F}_{IJKL}(\lambda^{Ii}\lambda^{Jj})(\lambda_i^K\lambda_j^L) , \tag{3.42}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_1 = & -2i\mathcal{F}\bar{\psi}_a^i\tilde{\sigma}^a\rho_i \\
& + \mathcal{F}_I \left((\tilde{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}}\tilde{\mathcal{D}}_a(\bar{\psi}_b^{\dot{\alpha}i}\bar{\lambda}_i^{I\dot{\beta}}) - \frac{1}{2}\tilde{\mathcal{D}}^a(\bar{\psi}_a^k\bar{\lambda}_k^I) \right. \\
& \quad - \frac{1}{2}\psi_{\alpha\dot{\alpha}\gamma k}(i\varepsilon^{\alpha\gamma}\bar{\kappa}^{\dot{\alpha}k}\bar{\phi}^I + i\varepsilon^{\alpha\gamma}S^{kl}\bar{\lambda}_l^{\dot{\alpha}} + i\varepsilon^{\alpha\gamma}\bar{W}^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{\dot{\beta}}^{Ik} - iY^{\alpha\gamma}\bar{\lambda}^{I\dot{\alpha}k}) \\
& \quad \left. - \frac{1}{2}\bar{\psi}_{\alpha\dot{\beta}\dot{\gamma}}^j(i\varepsilon^{\dot{\beta}\dot{\gamma}}\kappa_j^{\alpha}\bar{\phi}^I - 2i\varepsilon^{\dot{\beta}\dot{\gamma}}\rho_j^{\alpha}\phi^I + i\varepsilon^{\dot{\beta}\dot{\gamma}}S_{jk}\lambda^{I\alpha k} - i\bar{W}^{\dot{\beta}\dot{\gamma}}\lambda_j^{I\alpha} + i\varepsilon^{\dot{\beta}\dot{\gamma}}Y^{\alpha\beta}\lambda_{\beta j}^I) \right) \\
& + \mathcal{F}_{IJ} \left(\frac{1}{4}\varepsilon_{abcd}(\sigma^a)_{\alpha}^{\dot{\alpha}}f^{Ibc}\psi_k^{d\dot{\alpha}}\bar{\lambda}_{\dot{\alpha}}^{Jk} - \frac{1}{4}\varepsilon_{abcd}(\tilde{\sigma}^a)_{\dot{\alpha}}^{\alpha}f^{Ibc}\bar{\psi}_k^{d\dot{\alpha}}\lambda_{\alpha}^{Jk} + \frac{1}{2}\psi_{a\alpha}^j\lambda_j^{I\alpha}\tilde{\mathcal{D}}^a\bar{\phi}^J \right. \\
& \quad + (\sigma^{ab})_{\alpha}^{\beta}\psi_{a\beta}^j\lambda_j^{I\alpha}\tilde{\mathcal{D}}_b\bar{\phi}^J - \frac{1}{2}\bar{\psi}_{\alpha\dot{\alpha}\dot{\beta}}^j(-i\varepsilon^{\dot{\alpha}\dot{\beta}}S_{jk}\lambda^{I\alpha k}\phi^J + i\bar{W}^{\dot{\alpha}\dot{\beta}}\lambda_j^{I\alpha}\phi^J \\
& \quad \left. - i\varepsilon^{\dot{\alpha}\dot{\beta}}Y^{\alpha\beta}\lambda_{\beta j}^I\phi^J + i\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{S}_{jk}\lambda^{I\alpha k}\bar{\phi}^J - i\varepsilon^{\dot{\alpha}\dot{\beta}}W^{\alpha\beta}\lambda_{\beta j}^I\bar{\phi}^J + i\bar{Y}^{\dot{\alpha}\dot{\beta}}\lambda_j^{I\alpha}\bar{\phi}^J) \right) \\
& - \frac{i}{12}\mathcal{F}_{IJK}(\bar{\psi}_a^k\tilde{\sigma}^a\lambda^{Ij})(\lambda_j^K\lambda_k^J) , \tag{3.43}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_2 = & \mathcal{F} \bar{\psi}_{c\dot{\gamma}}^k \bar{\psi}_{d\dot{\delta}}^l (2(\tilde{\sigma}^{cd})^{\dot{\gamma}\dot{\delta}} S_{kl} - \varepsilon^{\dot{\gamma}\dot{\delta}} \varepsilon_{kl} (\tilde{\sigma}^{cd})_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\alpha}\dot{\beta}} - \varepsilon^{\dot{\gamma}\dot{\delta}} \varepsilon_{kl} (\sigma^{cd})_{\alpha\beta} Y^{\alpha\beta}) \\
& + \mathcal{F}_I \left(\frac{i}{2} (\sigma^{[a})_{\alpha}{}^{\dot{\alpha}} \psi^{b]}_{\dot{k}} \tilde{\mathcal{D}}_a \bar{\psi}_{b\dot{\alpha}}^k \bar{\phi}^I + \frac{i}{2} (\sigma^{[a})_{\alpha}{}^{\dot{\alpha}} \tilde{\mathcal{D}}_a \psi_{b\dot{k}}^{\alpha} \bar{\psi}^{b]}_{\dot{\alpha}}^k \bar{\phi}^I + \frac{1}{4} \varepsilon^{abcd} \tilde{\mathcal{D}}_b (\psi_{ck} \sigma_a \bar{\psi}_d^k) \bar{\phi}^I \right. \\
& - i(\psi_{ak} \sigma^b \bar{\psi}^{ak}) \tilde{\mathcal{D}}_b \bar{\phi}^I - i(\psi_{ak} \sigma^b \bar{\psi}_b^k) \tilde{\mathcal{D}}^a \bar{\phi}^I + i(\psi_{ak} \sigma^a \bar{\psi}_b^k) \tilde{\mathcal{D}}^b \bar{\phi}^I \\
& - \frac{1}{2} \varepsilon^{abcd} (\psi_{ak} \sigma_b \bar{\psi}_c^k) \tilde{\mathcal{D}}_d \bar{\phi}^I + \frac{1}{4} \psi_{c\dot{k}}^{\gamma} \psi_{d\dot{l}}^{\delta} (2(\sigma^{cd})_{\gamma\delta} S^{kl} \bar{\phi}^I + \varepsilon_{\gamma\delta} \varepsilon^{kl} (\tilde{\sigma}^{cd})_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\alpha}\dot{\beta}} \bar{\phi}^I \\
& + \varepsilon_{\gamma\delta} \varepsilon^{kl} (\sigma^{cd})_{\alpha\beta} Y^{\alpha\beta} \bar{\phi}^I) + \frac{1}{4} \bar{\psi}_{c\dot{\gamma}}^k \bar{\psi}_{d\dot{\delta}}^l (2(\tilde{\sigma}^{cd})^{\dot{\gamma}\dot{\delta}} \bar{S}_{kl} \bar{\phi}^I + \varepsilon^{\dot{\gamma}\dot{\delta}} \varepsilon_{kl} (\sigma^{cd})_{\beta\gamma} W^{\beta\gamma} \bar{\phi}^I \\
& + \varepsilon^{\dot{\gamma}\dot{\delta}} \varepsilon_{kl} (\tilde{\sigma}^{cd})_{\dot{\alpha}\dot{\beta}} \bar{Y}^{\dot{\alpha}\dot{\beta}} \bar{\phi}^I - 4(\tilde{\sigma}^{cd})^{\dot{\gamma}\dot{\delta}} S_{kl} \phi^I + 2\varepsilon^{\dot{\gamma}\dot{\delta}} \varepsilon_{kl} (\tilde{\sigma}^{cd})_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\alpha}\dot{\beta}} \phi^I \\
& + 2\varepsilon^{\dot{\gamma}\dot{\delta}} \varepsilon_{kl} (\sigma^{cd})_{\beta\gamma} Y^{\beta\gamma} \phi^I) + \frac{i}{2} \varepsilon^{abcd} \tilde{F}_{ab} \bar{\psi}_c \bar{\psi}_d \Big) \\
& + \frac{1}{4} \mathcal{F}_{IJ} \left(\psi_{a\dot{\alpha}}^j \bar{\psi}^{ak}_{\dot{\gamma}} \lambda^{I\alpha}_{\dot{j}} \bar{\lambda}^{J\dot{\gamma}}_k + (\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}} \psi_a^{\alpha} (\bar{\psi}_{bl})_{\dot{\alpha}} \lambda^{Ik}_{\alpha} \bar{\lambda}^{J\dot{\beta}}_l + (\sigma^{ab})_{\alpha\beta} \psi_a^{\alpha} (\bar{\psi}_{bl})_{\dot{\alpha}} \lambda^{I\beta k} \bar{\lambda}^{Jl}_{\dot{\alpha}} \right. \\
& + i\varepsilon_{abcd} \tilde{F}^{Iab} \psi^c \psi^d \bar{\phi}^J - \psi_{a\dot{k}}^{\alpha} \psi_{bl}^{\beta} \left(\frac{1}{2} (\sigma^{ab})_{\alpha\beta} \bar{\lambda}^{Ik} \bar{\lambda}^{Jl} + \frac{1}{4} \varepsilon_{\alpha\beta} \varepsilon^{kl} (\tilde{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \bar{\lambda}^{Ij}_{\dot{\alpha}} \bar{\lambda}^{Jj}_{\dot{\beta}} \right) \\
& \left. - i\varepsilon_{abcd} \tilde{F}^{Iab} \bar{\psi}^c \bar{\psi}^d \phi^J - \bar{\psi}_{c\dot{\gamma}}^k \bar{\psi}_{d\dot{\delta}}^l \left(\frac{1}{2} (\tilde{\sigma}^{cd})^{\dot{\gamma}\dot{\delta}} \lambda^I_k \lambda^J_l + \frac{1}{4} \varepsilon^{\dot{\gamma}\dot{\delta}} \varepsilon_{kl} (\sigma^{cd})_{\beta\gamma} \lambda^{I\beta i} \lambda^{J\gamma}_i \right) \right), \tag{3.44}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_3 = & \frac{1}{8} \mathcal{F}_{IJ} \left(\varepsilon^{abcd} (\bar{\lambda}^{Ik} \tilde{\sigma}_a \psi_{bk}) (\psi_c \psi_d) \bar{\phi}^J + \varepsilon^{abcd} (\lambda^{Ik} \sigma_a \bar{\psi}_{bk}) (\bar{\psi}_c \bar{\psi}_d) \phi^J \right. \\
& \left. - \varepsilon^{abcd} (\lambda^{Ik} \sigma_a \bar{\psi}_{bk}) (\psi_c \psi_d) \bar{\phi}^J - \varepsilon^{abcd} (\bar{\lambda}^{Ik} \tilde{\sigma}_a \psi_{bk}) (\bar{\psi}_c \bar{\psi}_d) \phi^J \right), \tag{3.45}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_4 = & -\frac{i}{4} \mathcal{F} \varepsilon^{abcd} (\bar{\psi}_a \bar{\psi}_b) (\bar{\psi}_c \bar{\psi}_d) \\
& + \mathcal{F}_I \left(\frac{i}{4} \varepsilon^{abcd} (\bar{\psi}_a \bar{\psi}_b) (\bar{\psi}_c \bar{\psi}_d) \phi^I - \frac{i}{8} \varepsilon^{abcd} (\psi_a \psi_b) (\bar{\psi}_c \bar{\psi}_d) \bar{\phi}^I \right) \\
& - \mathcal{F}_{IJ} \left(\frac{i}{16} \varepsilon^{abcd} (\psi_a \psi_b) (\psi_c \psi_d) \bar{\phi}^I \bar{\phi}^J + \frac{i}{16} \varepsilon^{abcd} (\bar{\psi}_a \bar{\psi}_b) (\bar{\psi}_c \bar{\psi}_d) \phi^I \phi^J \right. \\
& \left. - \frac{i}{8} \varepsilon^{abcd} (\psi_a \psi_b) (\bar{\psi}_c \bar{\psi}_d) \phi^I \bar{\phi}^J \right). \tag{3.46}
\end{aligned}$$

The use of torsion-free covariant derivatives $\tilde{\mathcal{D}}_a$ and the way we have grouped terms in the Lagrangian makes it easy to verify that in the case where $\mathcal{F}(\mathcal{W}) = \mathcal{W}^2$, the action is real.

One can check that the components S^{ij} , $Y_{\alpha\beta}$, $\rho_{\alpha i}$ and C intrinsic to $\text{SU}(2)$ superspace drop out of the action if we restrict to the superconformal case (*i.e.* \mathcal{F} of degree two). This is easy to understand. If we transform the action (3.33) to the manifestly superconformal framework, it becomes

$$S_c = \int d^4x d^4\theta \mathcal{E} \Phi^2 \mathcal{F}(\mathcal{W}^I / \Phi), \tag{3.47}$$

where Φ is the chiral compensator intrinsic to $\text{SU}(2)$ superspace. In the case where \mathcal{F} is of degree two, the dependence on the compensator vanishes and so the additional component fields associated with it must certainly vanish from the component action.

3.4 The tensor multiplet in components

As discussed in subsection 2.3, the tensor multiplet is described by a real primary superfield \mathcal{G}^{ij} of dimension 2 (2.26) satisfying the constraint (2.27). The corresponding 3-form field strength H in superspace is given by (2.24) and (2.25). Within the superfield \mathcal{G}^{ij} are found the matter components of the tensor multiplet: a real isotriplet field G^{ij} , a fermion $\chi_{\alpha i}$ and a complex scalar F ,

$$G^{ij} := \mathcal{G}^{ij}|, \quad (3.48a)$$

$$\chi_{\alpha i} := \frac{1}{3} \nabla_{\alpha}^j \mathcal{G}_{ij}|, \quad \bar{\chi}^{\dot{\alpha} i} := \frac{1}{3} \bar{\nabla}_{\dot{j}}^{\dot{\alpha}} \mathcal{G}^{ij}|, \quad (3.48b)$$

$$F := \frac{1}{12} \nabla^{ij} \mathcal{G}_{ij}|, \quad \bar{F} := \frac{1}{12} \bar{\nabla}^{ij} \mathcal{G}_{ij}|. \quad (3.48c)$$

The remaining component field, the two-form, is given by $b_{mn} := B_{mn}|$. Owing to the superspace identity

$$\nabla_{\alpha}^i \nabla_{\beta}^j \mathcal{G}^{kl} = -\frac{1}{6} \varepsilon_{\alpha\beta} \varepsilon^{i(k} \varepsilon^{l)j} \nabla^{pq} \mathcal{G}_{pq}, \quad (3.49)$$

there are no other independent component fields.

In constructing the three-form $H_{mnp} = 3\partial_{[m} B_{np]}$ we can make use of the superspace identity

$$H_{mnp} = E_m{}^A E_n{}^B E_p{}^C H_{ABC}(-)^{ab+ac+bc} \quad (3.50)$$

projected to lowest component. Defining

$$h_{mnp} := H_{mnp}| = 3\partial_{[m} b_{np]} \quad (3.51)$$

it is easy to show that

$$h_{mnp} = \varepsilon_{mnpq} \tilde{H}^a |e_a{}^q + 3i(\sigma_{[mn})_{\alpha}{}^{\beta} \psi_{p]}^{\alpha} \chi_{\beta}^j + 3i(\tilde{\sigma}_{[mn})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\psi}_{p]}^{\dot{\beta}} \bar{\chi}_{\dot{\alpha}}^j - 3(\sigma_{[m})_{\alpha\dot{\alpha}} \psi_n{}^{\alpha i} \bar{\psi}_{p]}^{\dot{\alpha} j} G_{ij} \quad (3.52)$$

or, equivalently,

$$\begin{aligned} \tilde{h}^a := \tilde{H}^a | &= \frac{1}{6} \varepsilon^{abcd} H_{bcd} = \frac{1}{2} \varepsilon^{abcd} \left(\frac{1}{3} h_{bcd} - i(\sigma_{cd})_{\alpha}{}^{\beta} \psi_b{}^{\alpha} \chi_c{}^{\beta} \right. \\ &\quad \left. - i(\tilde{\sigma}_{cd})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\psi}_b{}^{\dot{\beta}} \bar{\chi}_c{}^{\dot{\alpha}} + (\sigma_b)_{\alpha}{}^{\beta} \psi_c{}^{\alpha} \bar{\psi}_d{}^{\beta} G^k{}_l \right). \end{aligned} \quad (3.53)$$

We have emphasized the construction of the two-form multiplet completely geometrically, but it is worth noting that, as discussed in subsection 2.3, the two-form multiplet can be encoded in a chiral superfield Ψ . As shown in [40], by making use of the gauge transformations (2.29), one can choose all components of B_{AB} to vanish except for $B_{ab}|$ by imposing the component constraints⁹

$$\Psi| = 0, \quad \nabla_{\alpha}^i \Psi| = 0, \quad \nabla^{ij} \Psi| = \bar{\nabla}^{ij} \bar{\Psi}|. \quad (3.54)$$

One may easily construct b_{mn} using $b_{mn} = e_m{}^a e_n{}^b B_{ab}|$. As usual, the supersymmetry transformation laws of the component fields may be derived by using the constraints.

⁹The third constraint is not actually necessary to eliminate the other components of the two-form, but it does substantially simplify the component evaluation later.

3.5 Tensor multiplet action

The most general actions involving the self-couplings of tensor multiplets are naturally constructed in projective superspace. However, as was shown in [52] (see [53] for the curved superspace generalization), such actions can always be constructed from the chiral action¹⁰

$$S = \int d^4x d^4\theta \mathcal{E} \Psi \mathcal{W} + \text{c.c.} \quad (3.55)$$

where Ψ is the prepotential for the tensor multiplet and \mathcal{W} is the chiral field strength for some (possibly composite) vector multiplet. This action is invariant under the transformation (2.29) which ensures that only the physical components of the tensor multiplet appear. At the component level, it corresponds to a supersymmetric generalization of the $b \wedge f$ topological action, where f_{mn} is the two-form field strength of the vector multiplet \mathcal{W} . This model can describe self-couplings of tensor multiplets if a suitable composite vector multiplet can be constructed out of the tensor multiplets.

Using the gauge conditions on Ψ (3.54), the component action (3.13) is

$$S = \int d^4x e \left(F\phi + \chi_i^\alpha \lambda_\alpha^i + \frac{1}{8} G^{ij} X_{ij} - \frac{1}{4} \varepsilon^{mnpq} b_{mn} f_{pq} - \frac{i}{2} \bar{\psi}^{\dot{\alpha}\alpha i} (2\chi_{\alpha i} \phi + G_{ij} \lambda_\alpha^j) + (\tilde{\sigma}^{cd})^{\dot{\gamma}\delta} \bar{\psi}_{c\dot{\gamma}}^k \bar{\psi}_{d\dot{\delta}}^l G_{kl} \phi \right) + \text{c.c.} \quad (3.56)$$

This can be compared with the results in [40, 54, 55].

In the case of the improved tensor multiplet [55, 56], \mathcal{W} is a composite field, which we denote by \mathbb{W} :

$$\mathbb{W} = -\frac{1}{24\mathcal{G}} \bar{\nabla}_{ij} \mathcal{G}^{ij} + \frac{1}{36\mathcal{G}^3} \bar{\nabla}_{\dot{\alpha}k} \mathcal{G}^{ki} \bar{\nabla}_{\dot{l}}^{\dot{\alpha}} \mathcal{G}^{lj} \mathcal{G}_{ij} , \quad (3.57)$$

where $\mathcal{G}^2 = \frac{1}{2} \mathcal{G}^{ij} \mathcal{G}_{ij}$. This multiplet was first reconstructed in curved superspace in [40], based on the results of [52, 55]. The components ϕ , λ_α^i , and X_{ij} of this composite vector

¹⁰This action in superspace appeared in [46] for the flat case, and was generalized to curved superspace in [40]. Its component form, coupled to conformal supergravity, was given in [54, 55]. A gauge fixing relates this to the component linear multiplet Lagrangian constructed in [42].

multiplet are given by

$$\phi = \mathcal{W}| = -\frac{1}{2G}\bar{F} + \frac{1}{4G^3}\bar{\chi}^i\bar{\chi}^j G_{ij} , \quad (3.58a)$$

$$\begin{aligned} \lambda_\alpha^i = \nabla_\alpha^i \mathcal{W}| &= \frac{1}{G}(\mathrm{i}\nabla_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}i} - W_{\alpha\beta}\chi^{\beta i} - 3\Sigma_{\alpha j}G^{ij}) \\ &+ \frac{1}{2G^3}(\bar{F}\chi_{\alpha j}G^{ij} - \mathrm{i}\tilde{h}_{\alpha\dot{\alpha}}\bar{\chi}_j^{\dot{\alpha}}G^{ij} - \mathrm{i}\nabla_{\alpha\dot{\alpha}}G^{ij}\bar{\chi}^{\dot{\alpha}k}G_{jk} + \chi_{\alpha j}\bar{\chi}_{\dot{\alpha}}^i\bar{\chi}^{\dot{\alpha}j}) \\ &- \frac{3}{4G^5}\chi_{\alpha j}\bar{\chi}_{\dot{\alpha}}^k\bar{\chi}^{\dot{\alpha}l}G_{kl}G^{ij} , \end{aligned} \quad (3.58b)$$

$$\begin{aligned} X^{ij} = \nabla^{ij} \mathcal{W}| &= \frac{1}{G}\left(-2\Box G^{ij} - 6G^{ij}D + 6\bar{\Sigma}_{\dot{\beta}}^{(i}\bar{\chi}^{\dot{\beta}j)} + 6\Sigma^{\beta(i}\chi_{\beta}^{j)}\right) \\ &+ \frac{1}{G^3}\left(-6\Sigma^{\beta k}\chi_{\beta}^l G_{kl}G^{ij} - 6\bar{\Sigma}_{\dot{\beta}}^k\bar{\chi}^{\dot{\beta}l}G_{kl}G^{ij} - W^{\alpha\beta}\chi_{\alpha}^k\chi_{\beta k}G^{ij} + \bar{W}_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\alpha}k}\bar{\chi}_{\dot{\beta}k}G^{ij}\right. \\ &\quad - 2\mathrm{i}\bar{\chi}^{\dot{\alpha}k}G_k^{(i}\nabla_{\alpha\dot{\alpha}}\chi^{\alpha j)} - 2\mathrm{i}\chi_k^{\alpha}G^{k(i}\nabla_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}j)} + \nabla^a G^{ik}\nabla_a G^{jl}G_{kl} \\ &\quad + \tilde{h}^a\tilde{h}_a G^{ij} + G^{ij}F\bar{F} + \bar{\chi}_{\dot{\alpha}}^i\bar{\chi}^{\dot{\alpha}j}F + \chi^{\alpha i}\chi_{\alpha}^j\bar{F} - 2\tilde{h}^a\nabla_a G^{k(i}G^{j)k} \\ &\quad \left.+ \mathrm{i}\tilde{h}^{\alpha\dot{\alpha}}\chi_{\alpha}^{(i}\bar{\chi}_{\dot{\alpha}}^{j)} - \frac{\mathrm{i}}{2}\nabla^{\alpha\dot{\alpha}}G^{k(i}\chi_{\alpha k}\bar{\chi}_{\dot{\alpha}}^{j)} - \frac{\mathrm{i}}{2}\nabla^{\alpha\dot{\alpha}}G^{k(i}\chi_{\alpha}^{j)}\bar{\chi}_{\dot{\alpha}k}\right) \\ &+ \frac{1}{G^5}\left(-\frac{3}{2}F G^{ij}\bar{\chi}_{\dot{\alpha}}^k\bar{\chi}^{\dot{\alpha}l}G_{kl} - \frac{3}{2}\bar{F} G^{ij}\chi^{\alpha k}\chi_{\alpha}^l G_{kl} + 3\mathrm{i}\tilde{h}_{\alpha\dot{\alpha}}\chi_{\alpha}^k\bar{\chi}_{\dot{\alpha}}^l G^{kl}G^{ij}\right. \\ &\quad - 3\mathrm{i}\nabla_{\alpha\dot{\alpha}}G^{k(i}G^{j)l}\chi_{(l}\bar{\chi}_{\dot{\alpha}}^{q)}G_k^q - 3\chi^{\alpha i}\chi_{\alpha}^j\bar{\chi}_{\dot{\alpha}}^k\bar{\chi}^{\dot{\alpha}l}G_{kl} \\ &\quad \left.- 3\chi^{\alpha k}\chi_{\alpha}^l\bar{\chi}_{\dot{\alpha}}^i\bar{\chi}^{\dot{\alpha}j}G_{kl} + 3\chi_k^{\alpha}\chi_{\alpha}^{(i}\bar{\chi}_{\dot{\alpha}}^{j)}\bar{\chi}_{\dot{\alpha}}^{kl}\right) \\ &+ \frac{15}{4G^7}G^{ij}\chi^{\alpha k}\chi_{\alpha}^l G_{kl}\bar{\chi}_{\dot{\alpha}}^p\bar{\chi}^{\dot{\alpha}q}G_{pq} , \end{aligned} \quad (3.58c)$$

where

$$\nabla_a G_{ij} \equiv \nabla'_a G^{ij} - \psi_a^{\gamma(i}\chi_{\gamma}^{j)} + \psi_a^{\dot{\gamma}(i}\bar{\chi}_{\dot{\gamma}}^{j)} , \quad (3.59a)$$

$$\nabla_a \bar{\chi}^{\dot{\alpha}i} \equiv \nabla'_a \bar{\chi}^{\dot{\alpha}i} + \frac{\mathrm{i}}{2}\psi_a^{\gamma i}\tilde{h}_{\gamma}^{\dot{\alpha}} + \frac{\mathrm{i}}{2}\psi_a^{\gamma}\nabla_{\gamma}^{\dot{\alpha}}G^{ik} + \frac{1}{2}\bar{\psi}_a^{\dot{\alpha}i}\bar{F} + 2\phi_{a j}^{\dot{\alpha}}G^{ij} , \quad (3.59b)$$

$$\begin{aligned} \Box G^{ij} &\equiv \frac{1}{2}\nabla'_a \nabla^a G^{ij} - \psi_a^{\alpha(i}\nabla_{\alpha}^{j)}\chi_{\alpha}^{j)} - \frac{\mathrm{i}}{2}\psi^{\alpha\dot{\alpha}(i}\bar{W}_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\beta}j)} - \frac{3\mathrm{i}}{4}\psi^{\alpha\dot{\alpha}}\chi_{\alpha k}\bar{\Sigma}_{\dot{\alpha}}^k G^{ij} \\ &+ \frac{3\mathrm{i}}{4}\psi^{\alpha\dot{\alpha}(i}\bar{\Sigma}_{\dot{\alpha}k}G^{j)k} + \frac{3\mathrm{i}}{4}\psi^{\alpha\dot{\alpha}}\chi_{\alpha k}\bar{\Sigma}_{\dot{\alpha}}^{(i}G^{j)k} \\ &+ 2\mathfrak{f}_a^{\alpha}G^{ij} - \mathrm{i}\phi_{\alpha\dot{\alpha}}\chi^{\alpha(i}\bar{\chi}^{\dot{\alpha}j)} + \text{c.c.} \end{aligned} \quad (3.59c)$$

There is one additional component field in the composite vector multiplet: the gauge connection v_m . Its field strength f_{mn} is contained within the supercovariant field strength \hat{F}_{ab} (3.20), which possesses the self-dual part

$$\begin{aligned} \hat{F}_{\alpha\beta} &= -\frac{1}{8}\nabla_{\alpha\beta}\mathcal{W}| - \frac{1}{2}W_{\alpha\beta}\bar{\mathcal{W}}| \\ &= -\frac{1}{4}\nabla_{(\alpha\dot{\alpha}}\left(\frac{1}{\mathcal{G}}\tilde{H}_{\beta)}^{\dot{\alpha}} - \frac{\mathrm{i}}{9\mathcal{G}^3}\nabla_{\beta)}^k\mathcal{G}_k^{i\dot{\alpha}}\bar{\nabla}_l^{\dot{\alpha}}\mathcal{G}^{lj}G_{ij}\right) + \frac{1}{16G^3}\nabla_{\alpha\dot{\alpha}}G_k^{i\dot{\alpha}}\nabla_{\beta}^{\dot{\alpha}}G^{jk}G_{ij} \\ &+ \frac{1}{8G}\chi_k^{\gamma}\nabla_{\gamma}^k W_{\alpha\beta}| - \frac{1}{16G}G_{ij}\nabla^{ij}W_{\alpha\beta}| . \end{aligned} \quad (3.60)$$

This is a rather complicated result. The last two terms must be evaluated to lowest components, while the remaining terms involve a supercovariant derivative ∇_a . Evaluating this explicitly requires the identities

$$\begin{aligned} K_a \tilde{H}_b &= 0, \quad S_k^\gamma \tilde{H}_a| = -3i(\sigma_a)^{\gamma\dot{\alpha}} \bar{\chi}_{\dot{\alpha}k}, \quad \bar{S}_{\dot{\gamma}}^k \tilde{H}_a| = 3i(\sigma_a)_{\alpha\dot{\gamma}} \chi^{\alpha k}, \\ \nabla_\alpha^i \tilde{H}_{\beta\dot{\beta}}| &= -\nabla_{\alpha\dot{\beta}} \chi_\beta^i + \varepsilon_{\alpha\beta} \nabla_{\gamma\dot{\beta}} \chi^{\gamma i} + i\varepsilon_{\alpha\beta} \bar{W}_{\dot{\beta}\dot{\gamma}} \bar{\chi}^{\dot{\gamma}i} + 3\varepsilon_{\alpha\beta} \bar{\Sigma}_{\dot{\beta}\dot{\gamma}} G^{ij}. \end{aligned} \quad (3.61)$$

However, it is easiest to specify the field strength f_{mn} directly and reconstruct the supercovariant two-form from (3.20) if needed. Using the results of section 4, we will show that¹¹

$$f_{mn} = 2\partial_{[m}\Gamma_{n]} + \frac{1}{4G^3} \partial_m G^{ik} \partial_n G_k^j G_{ij}, \quad (3.62a)$$

$$\Gamma_m := \frac{1}{2G} \phi_m^{ij} G_{ij} + \frac{1}{2G} e_m^a \tilde{h}_a + \frac{1}{4G} (\psi_m^j \chi_j) + \frac{1}{4G} (\bar{\psi}_m^j \bar{\chi}_j) + \frac{i}{4G^3} (\chi^i \sigma_m \bar{\chi}^j) G_{ij}. \quad (3.62b)$$

As observed in [55], there is no $SU(2)$ covariant expression for the composite one-form v_m from which we could construct f_{mn} . Although the first term in f_{mn} is exact, the second is not. We will see in the next section an explanation within projective superspace for this behavior.¹²

4 Projective superspace and component reductions

Recently, a projective superspace formulation of conformal supergravity has been constructed [10]. Soon afterwards, it was shown how to perform component reductions in this framework [58], but to our knowledge, this component reduction has not been applied to any specific action yet. In this section, we will show how this can be done explicitly. By first constructing the projective superspace analogue of the action (3.55), we will show how to use the component reduction formula in projective superspace to reproduce the component result (3.56).

The construction of [10] was based on [34]; we will begin by generalizing it to the conformal superspace geometry developed in [14]. After briefly describing the formulation of the vector and tensor multiplets in projective superspace, we will proceed to the component reduction of the projective action analogous to (3.55).

4.1 Formalism of projective superspace

Following [10] the supermanifold $\mathcal{M}^{4|8}$ is augmented with an additional \mathbb{CP}^1 parametrized by an isotwistor coordinate $v^i \in \mathbb{C}^2 \setminus \{0\}$. Matter fields are constructed in terms of covariant projective multiplets $\mathcal{Q}^{(n)}(z, v)$ which are holomorphic in the isotwistor v^i and of definite homogeneity, $\mathcal{Q}^{(n)}(z, cv) = c^n \mathcal{Q}^{(n)}(z, v)$, on an open domain of $\mathbb{C}^2 \setminus \{0\}$. Such superfields are intrinsically defined on \mathbb{CP}^1 .

¹¹The second term in (3.62a) transforms non-covariantly under $SU(2)$ transformations, but this is cancelled by the transformation of the explicit $SU(2)$ connection in Γ_m .

¹²Within harmonic superspace, this feature and its origin were discussed in the rigid supersymmetric case [5, 57].

Using the isotwistor coordinate, one may construct a subset of anti-commuting spinor derivatives¹³

$$\nabla_\alpha^+ := v_i \nabla_\alpha^i, \quad \bar{\nabla}_{\dot{\alpha}}^+ := v_i \bar{\nabla}_{\dot{\alpha}}^i, \quad \{\nabla_\alpha^+, \bar{\nabla}_{\dot{\alpha}}^+\} = \{\nabla_\alpha^+, \bar{\nabla}_\beta^+\} = 0. \quad (4.1)$$

The $+$ on these operators denotes that they are of degree $+1$ in v^i . Projective superfields are required to be analytic with respect to these derivatives,

$$\nabla_\alpha^+ \mathcal{Q}^{(n)} = \bar{\nabla}_{\dot{\alpha}}^+ \mathcal{Q}^{(n)} = 0. \quad (4.2)$$

Since these superfields depend on only half of the Grassmann coordinates of superspace, they play the same fundamental role in $\mathcal{N} = 2$ supersymmetric theories as chiral superfields play in $\mathcal{N} = 1$. Using such fields, one may construct an action principle [10]

$$S = \frac{1}{2\pi} \oint_C v^i dv_i \int d^4x d^4\theta d^4\bar{\theta} E \frac{\mathcal{W}_0 \bar{\mathcal{W}}_0}{(\Sigma_0^{++})^2} \mathcal{L}^{++}, \quad E = \text{Ber}(E_M^A), \quad (4.3)$$

with $\Sigma_0^{++} := (\nabla^+)^2 \mathcal{W}_0 / 4 = (\bar{\nabla}^+)^2 \bar{\mathcal{W}}_0 / 4$. The Lagrangian $\mathcal{L}^{++}(z, v)$ is a covariant real projective multiplet of weight-two. The vector multiplet \mathcal{W}_0 is used here essentially as a compensator, and the form of the action simplifies if the gauge $\mathcal{W}_0 = 1$ is taken (or, equivalently, if all objects are redefined in terms of \mathcal{W}_0 as in section 2.4). This action appears to depend on the choice of \mathcal{W}_0 , but one can show that this is purely illusory. The role of the compensator is simply to allow us to write the action over the full set of superspace coordinates.

As discussed in [10], it is useful in the evaluation of this action to introduce an additional fixed isotwistor u_i which obeys $v^i u_i \neq 0$ along the contour of integration. In terms of v^i and u_i , one may introduce the objects¹⁴

$$u^{i+} = v^i, \quad u_i^- = \frac{u_i}{v^k u_k}, \quad (4.4)$$

which obey $u^{i+} u_i^- = 1$. Associated with these are derivative operations

$$D^{++} = (v^j u_j) v_i \frac{\partial}{\partial u_i}, \quad D^0 = v^i \frac{\partial}{\partial v^i} - u_i \frac{\partial}{\partial u_i}, \quad D^{--} = \frac{1}{(v^j u_j)} u^i \frac{\partial}{\partial v^i}. \quad (4.5)$$

The charges on the derivative operators and on u_i^\pm denote their homogeneity in v^i . Each is defined to be of degree *zero* in u_i . All other operators and fields which we introduce will similarly be degree zero in u_i but of some fixed homogeneity in v^i .

We should note that the derivative operators may be defined formally as

$$D^{++} = u_i^+ \frac{\partial}{\partial u_i^-}, \quad D^0 = u^{i+} \frac{\partial}{\partial u^{i+}} - u_i^- \frac{\partial}{\partial u_i^-}, \quad D^{--} = u^{i-} \frac{\partial}{\partial u^{i+}}, \quad (4.6)$$

¹³Within [10], $\text{SU}(2)$ superspace and the corresponding covariant derivatives \mathcal{D}_A were used to construct projective multiplets, but the generalization to the superconformal covariant derivative ∇_A is entirely straightforward.

¹⁴In [10], u_i^- was defined as u_i so that $u^{i+} u_i^- \neq 1$. We find it more convenient to normalize u_i^- .

acting on the variables u_i^{i+} and u_i^- . Fields and operators of definite homogeneity in v^i can be reinterpreted as possessing definite D^0 charge. Written in this form, the operators superficially resemble the corresponding objects defined in harmonic superspace [5]. For this reason we will frequently refer to u_i^\pm as “harmonics”, but their real origin should be kept in mind.

Just as superspace geometrizes supersymmetry, the auxiliary \mathbb{CP}^1 geometrizes isospin. The components q^i of an isospinor are naturally associated with a weight-one isotwistor $q^+ = q^i u_i^+$. The operator $\lambda^{jk} J_{jk}$ acts on q^i as $\lambda^{jk} J_{jk} q^i = \lambda^i_j q^j$. In order to mimic this action on q^+ , we identify

$$J_{ij} = -u_i^+ u_j^+ D^{--} + u_{(i}^+ u_{j)}^- D^0 + u_i^- u_j^- D^{++} , \quad (4.7)$$

leading to

$$\lambda^{ij} J_{ij} q^+ = (-\lambda^{++} D^{--} + \lambda^{+-} D^0 + \lambda^{--} D^{++}) q^+ = -\lambda^{++} q^- + \lambda^{+-} q^+ = \lambda^+_j q^j , \quad (4.8)$$

where we have denoted $\lambda^{++} = \lambda^{ij} u_i^+ u_j^+$, $\lambda^{+-} = \lambda^{ij} u_i^+ u_j^-$ and so on. The above definition similarly works for q^- , as well as for objects with any number of $SU(2)$ indices.

This construction becomes nontrivial when we consider *general* functions $\mathcal{Q}^{(n)}$ on \mathbb{CP}^1 of fixed homogeneity which are defined only in certain regions. They need not be polynomial in v^i and furnish infinite dimensional representations of $SU(2)$. Such functions are called “arctic” if they are well-defined in the northern chart of \mathbb{CP}^1 and “antarctic” if they are well-defined in the southern chart. An arctic multiplet $\Upsilon^{(n)}$ is necessarily complex, and is related to an antarctic multiplet $\check{\Upsilon}^{(n)}$ by the smile conjugation operation [10].

Along with the operators ∇_α^+ and $\bar{\nabla}_{\dot{\alpha}}^+$, we introduce ∇_α^- and $\bar{\nabla}_{\dot{\alpha}}^-$ constructed using u_i^- . Their algebra is easily found by contracting the harmonics u_i^\pm with eq. (2.10). Of particular interest is the isotwistor reformulation of the algebra of S_i^α with ∇_β^j :

$$\{S_\alpha^\pm, \nabla_\beta^\pm\} = \pm 4\varepsilon_{\alpha\beta} D^{\pm\pm}, \quad \{\bar{S}^{\dot{\alpha}\pm}, \bar{\nabla}^{\dot{\beta}\pm}\} = \mp 4\varepsilon^{\dot{\alpha}\dot{\beta}} D^{\pm\pm} , \quad (4.9a)$$

$$\{S_\alpha^\mp, \nabla_\beta^\pm\} = \pm(2\varepsilon_{\alpha\beta} \mathbb{D} - 4M_{\alpha\beta} - \varepsilon_{\alpha\beta} Y) - 2\varepsilon_{\alpha\beta} D^0 , \quad (4.9b)$$

$$\{\bar{S}^{\dot{\alpha}\mp}, \bar{\nabla}^{\dot{\beta}\pm}\} = \mp(2\varepsilon^{\dot{\alpha}\dot{\beta}} \mathbb{D} + 4\bar{M}^{\dot{\alpha}\dot{\beta}} + \varepsilon^{\dot{\alpha}\dot{\beta}} Y) + 2\varepsilon^{\dot{\alpha}\dot{\beta}} D^0 . \quad (4.9c)$$

These are quite powerful equations. They tell us that if a superfield $\mathcal{Q}^{(n)}$ obeys the analyticity condition (4.2) and is also primary, then it must obey the further conditions implied by $\{S_\beta^+, \nabla_\alpha^+\} \mathcal{Q}^{(n)} = \{S_\beta^-, \nabla_\alpha^+\} \mathcal{Q}^{(n)} = 0$ and their complex conjugates. These conditions lead to¹⁵

$$Y \mathcal{Q}^{(n)} = 0 , \quad \mathbb{D} \mathcal{Q}^{(n)} = D^0 \mathcal{Q}^{(n)} = n \mathcal{Q}^{(n)} , \quad D^{++} \mathcal{Q}^{(n)} = 0 . \quad (4.10)$$

In particular, the last condition implies that $\mathcal{Q}^{(n)}$ must be independent of u_i .

An important property of the contour integral is the vanishing of a total D^{--} derivative,

$$\oint v^i dv_i D^{--} \mathcal{F} = 0 \quad (4.11)$$

¹⁵These relations were first discussed by Kuzenko for the case of globally superconformal multiplets in 4D [59], extending the work in 5D [60]. The generalization to the local case appeared in [10].

where \mathcal{F} is an arbitrary weight-zero function of v^i and u_i . A simple proof is given in appendix D of [53].

4.2 Component reductions in projective superspace

The reduction of the general projective action (4.3) to components was carried out in [58] using SU(2) superspace. It is straightforward to convert the result given there to one involving the superconformally covariant derivative ∇_A . The action (4.3) reduces to

$$S = \frac{1}{2\pi} \oint u^{i+} du_i^+ \int d^4x e \mathcal{L}^{--}, \quad e = \det(e_m^a), \quad (4.12)$$

where the isotwistor-dependent component Lagrangian \mathcal{L}^{--} depends on the auxiliary isotwistor u_i in the combination $u_i^- = u_i/(v^k u_k)$. We have written the contour measure in terms of u^{i+} to emphasize its degree of homogeneity in v^i . As discussed in [58], the action is independent of the choice of the isotwistor u_i ; it is subject only to the requirement that $v^k u_k \neq 0$ along the integration contour.

The full expression for the component Lagrangian is

$$\begin{aligned} \mathcal{L}^{--} = & \frac{1}{16} (\nabla^-)^2 (\bar{\nabla}^-)^2 \mathcal{L}^{++} - \frac{i}{8} (\bar{\psi}_m^- \tilde{\sigma}^m)^\alpha \nabla_\alpha^- (\bar{\nabla}^-)^2 \mathcal{L}^{++} - \frac{i}{8} (\psi_m^- \sigma^m)_{\dot{\alpha}} \bar{\nabla}^{\dot{\alpha}-} (\nabla^-)^2 \mathcal{L}^{++} \\ & - \frac{1}{4} \left((\psi_m^- \sigma^{mn})^\alpha \bar{\psi}_n^- \dot{\alpha}- + \psi_m^- \alpha- (\tilde{\sigma}^{mn} \bar{\psi}_n^-)^{\dot{\alpha}} - i \phi_m^{--} (\sigma^m)^{\dot{\alpha}\alpha} \right) [\nabla_\alpha^-, \bar{\nabla}_{\dot{\alpha}}^-] \mathcal{L}^{++} \\ & - \frac{1}{4} (\psi_m^- \sigma^{mn} \psi_n^-) (\nabla^-)^2 \mathcal{L}^{++} - \frac{1}{4} (\bar{\psi}_m^- \tilde{\sigma}^{mn} \bar{\psi}_n^-) (\bar{\nabla}^-)^2 \mathcal{L}^{++} \\ & - \left(\frac{1}{2} \varepsilon^{mnpq} (\psi_m^- \sigma_n \bar{\psi}_p^-) \psi_q^- \alpha- - 2 (\psi_m^- \sigma^{mn})^\alpha \phi_n^{--} \right) \nabla_\alpha^- \mathcal{L}^{++} \\ & + \left(\frac{1}{2} \varepsilon^{mnpq} (\bar{\psi}_m^- \tilde{\sigma}_n \psi_p^-) \bar{\psi}_{q\dot{\alpha}}^- - 2 (\bar{\psi}_m^- \tilde{\sigma}^{mn})_{\dot{\alpha}} \phi_n^{--} \right) \bar{\nabla}^{\dot{\alpha}-} \mathcal{L}^{++} \\ & - 3 \varepsilon^{mnpq} (\psi_m^- \sigma_n \bar{\psi}_p^-) \phi_q^{--} \mathcal{L}^{++}, \end{aligned} \quad (4.13)$$

where projection to components is implicitly applied to all covariant spinor derivatives of \mathcal{L}^{++} . The auxiliary isotwistor u_i appears explicitly in the expressions

$$\nabla_\alpha^- := u_i^- \nabla_\alpha^i, \quad \bar{\nabla}_{\dot{\alpha}}^- := u_i^- \bar{\nabla}_{\dot{\alpha}}^i, \quad (4.14)$$

$$\psi_{m\alpha}^- := u_i^- \psi_{m\alpha}^i, \quad \bar{\psi}_{m\dot{\alpha}}^- := u_i^- \bar{\psi}_{m\dot{\alpha}}^i, \quad \phi_m^{--} := u_i^- u_j^- \phi_m^{ij}, \quad (4.15)$$

as well as implicitly in the definition of the SU(2) connection within the covariant derivative. Note that in addition to the gravitino, the SU(2) connection ϕ_m^{--} appears *explicitly* in the action, and for much the same reason: they both correspond to connections on an auxiliary manifold which is integrated over – a Grassmann manifold in the case of the gravitino and a \mathbb{CP}^1 manifold in the case of the isospin connection.

In later sections, we will also use the following notations:

$$\psi_{m\alpha}^+ := u_i^+ \psi_{m\alpha}^i, \quad \bar{\psi}_{m\dot{\alpha}}^+ := u_i^+ \bar{\psi}_{m\dot{\alpha}}^i. \quad (4.16)$$

4.3 Abelian vector and tensor multiplets in projective superspace

Projective superspace provides a natural realization for both the $\mathcal{N} = 2$ abelian vector multiplet and the $\mathcal{N} = 2$ tensor multiplet. We briefly review their constructions here.

The tensor multiplet \mathcal{G}^{ij} is naturally associated with a projective multiplet \mathcal{G}^{++} of weight-two, which is required to be defined everywhere on \mathbb{CP}^1 . This last condition implies that $\mathcal{G}^{++} = \mathcal{G}^{ij} u_i^+ u_j^+$ and then the analyticity condition (4.2) becomes equivalent to (2.27).

The abelian vector multiplet structure is a bit more intricate. Recall that it is described by a chiral superfield \mathcal{W} obeying the Bianchi identity (2.18). The chirality condition and the Bianchi identity are naturally satisfied by the definition

$$\mathcal{W} := \frac{1}{8\pi} \oint u^{i+} du_i^+ (\bar{\nabla}^-)^2 \mathcal{V} . \quad (4.17)$$

\mathcal{V} is a real weight-zero projective multiplet, which serves as a prepotential for the $\mathcal{N} = 2$ abelian vector multiplet. The reduced chirality of this expression was demonstrated in [61, 62] using $SU(2)$ superspace, but the extension to conformal superspace is straightforward. The prepotential \mathcal{V} possesses a gauge transformation

$$\mathcal{V} \rightarrow \mathcal{V} + \Lambda + \check{\Lambda} , \quad (4.18)$$

where Λ is an arctic multiplet of weight zero and $\check{\Lambda}$ is its smile conjugate. One can check that \mathcal{W} is invariant under this transformation [61, 62]. In the Minkowski case, one can further show that \mathcal{V} decomposes into separate prepotentials respectively for the $\mathcal{N} = 1$ abelian vector and chiral multiplets, along with an infinite set of unconstrained $\mathcal{N} = 1$ multiplets which are pure gauge degrees of freedom [9].

4.4 Component evaluation

General superconformal two-derivative actions involving several tensor multiplets \mathcal{G}_A^{++} are naturally constructed in projective superspace by choosing a Lagrangian \mathcal{L}^{++} which is an analytic function of \mathcal{G}_A^{++} of degree one. It was shown in [52, 53] how to relate this to the formulation (3.55). Such models of several tensor multiplets coupled to conformal supergravity were also considered at the component level in [63].

For a single tensor multiplet, there is a unique action in projective superspace known as the improved tensor multiplet action,¹⁶

$$\mathcal{L}^{++} = \mathcal{G}^{++} \ln(\mathcal{G}^{++} / i\Upsilon^+ \check{\Upsilon}^+) , \quad (4.19)$$

where Υ^+ is a weight-one arctic multiplet. As discussed in [58, 61], the appearance of the arctic multiplet in the action has no effect on the physics.¹⁷ Under the redefinition $\Upsilon^+ \rightarrow \Upsilon^+ e^{-\Lambda}$ where Λ is a weight-zero arctic multiplet, the action changes as $\mathcal{L}^{++} \rightarrow$

¹⁶ The improved tensor multiplet action coupled to supergravity appeared for the first time in [61]. The flat analogue appeared originally in [6].

¹⁷ This action resembles the $\mathcal{N} = 1$ improved tensor action, $\mathcal{L} = G \ln(G/\Phi\bar{\Phi})$. The multiplet Φ is chiral but has no physical effect on the action.

$\mathcal{L}^{++} + \mathcal{G}^{++}(\Lambda + \check{\Lambda})$. This additional term was shown in [58] to vanish by inspection of the component action.

Given this Lagrangian, we may immediately apply the component reduction formula (4.13). In principle, this is a straightforward process and for superconformal tensor multiplet models the contour integral can (at least in principle) be performed since there are only a finite number of degrees of freedom.

Here we take an approach analogous to (3.55). The Lagrangian (4.19) can be understood as

$$\mathcal{L}^{++} = \mathcal{G}^{++}\mathcal{V} , \quad (4.20)$$

where \mathcal{V} is a weight-zero projective multiplet. The gauge invariance $\mathcal{V} \rightarrow \mathcal{V} + \Lambda + \check{\Lambda}$ allows the identification of \mathcal{V} as the projective prepotential for an abelian vector multiplet. In this case, it is a *composite* vector multiplet prepotential and completely equivalent to the action (3.55).

In this section, we undertake the evaluation of the action (4.20) using the component Lagrangian (4.13), so that the comparison to (3.56) is completely clear. That is, we will organize terms in such a way that the explicit contour integral is eliminated, with the resulting component expression depending only on \mathcal{G}^{ij} and the physical components of \mathcal{V} . Only then will we identify \mathcal{V} as a composite multiplet. This will serve as a useful test for the component reduction rule (4.13), which can be used in principle for cases (such as models with arctic multiplets) where other methods of component reduction are prohibitively difficult (or non-existent). Along the way, we will also discover a useful result about how to construct the one-form v_m from the projective prepotential \mathcal{V} .

This is a nontrivial calculation involving a number of integrations by parts. It is also, to our knowledge, the first *direct* application of the component reduction rule (4.13). For this reason, we will present a detailed summary of the calculation. First, we will present the full calculation in the *globally* superconformal case to emphasize the ideas which are important in organizing the action. Then we will present a summary of the locally superconformal calculation emphasizing the techniques which are needed to evaluate it fully.

4.4.1 Globally superconformal action

In evaluating the component Lagrangian (4.13) corresponding to the projective Lagrangian (4.20), there is a single term in the absence of conformal supergravity:

$$\mathcal{L}^{--} = \frac{1}{16}(D^-)^2(\bar{D}^-)^2(\mathcal{G}^{++}\mathcal{V}) . \quad (4.21)$$

Nevertheless, its evaluation is not entirely trivial. One begins by constructing a number of identities for the $\mathcal{N} = 2$ tensor multiplet:

$$(\bar{D}^-)^2\mathcal{G}^{++} = (\bar{D}^+)^2\mathcal{G}^{--} , \quad (4.22a)$$

$$D_\alpha^-(\bar{D}^-)^2\mathcal{G}^{++} = D_\alpha^-(\bar{D}^+)^2\mathcal{G}^{--} = 4i\partial_{\alpha\dot{\beta}}\bar{D}^{\dot{\beta}+}\mathcal{G}^{--} , \quad (4.22b)$$

$$(D^-)^2(\bar{D}^-)^2\mathcal{G}^{++} = (D^-)^2(\bar{D}^+)^2\mathcal{G}^{--} = 16\Box\mathcal{G}^{--} . \quad (4.22c)$$

In addition, using the components of the tensor multiplet as defined in section 3.4, one can show that

$$D_\alpha^- \mathcal{G}^{++}| = -2\chi_\alpha^+ , \quad (4.23a)$$

$$(\bar{D}^-)^2 \mathcal{G}^{++}| = 4\bar{F} , \quad (4.23b)$$

$$D_\alpha^- \bar{D}_{\dot{\beta}}^- \mathcal{G}^{++}| = -2i\partial_{\alpha\dot{\beta}} G^{+-} + 2i\tilde{H}_{\alpha\dot{\beta}} , \quad (4.23c)$$

with the obvious definitions $\chi_\alpha^+ = \chi_\alpha^i u_i^+$ and $G^{+-} = G^{ij} u_i^+ u_j^-$. In this section, we will abuse notation and frequently use χ_α^+ , \bar{F} , $\tilde{H}_{\alpha\dot{\alpha}}$, and so on to describe both the component fields as well as the superfields with these corresponding lowest components.

Applying these rules, it is straightforward to show that

$$\begin{aligned} \mathcal{L}^{--} = & \square \mathcal{G}^{--} \mathcal{V} + i\partial_{\alpha\dot{\beta}} \bar{\chi}^{\dot{\beta}-} D^{\alpha-} \mathcal{V} + i\partial_{\alpha\dot{\beta}} \chi^{\alpha-} D^{\dot{\beta}-} \mathcal{V} + \frac{1}{4} F (\bar{D}^-)^2 \mathcal{V} + \frac{1}{4} \bar{F} (D^-)^2 \mathcal{V} \\ & - \frac{i}{2} \partial_{\alpha\dot{\alpha}} \mathcal{G}^{--} D^{\alpha-} \bar{D}^{\dot{\alpha}-} \mathcal{V} + \frac{i}{2} \tilde{H}_{\alpha\dot{\alpha}} D^{\alpha-} \bar{D}^{\dot{\alpha}-} \mathcal{V} - \frac{1}{4} \chi^{\alpha+} D_\alpha^- (\bar{D}^-)^2 \mathcal{V} + \frac{1}{4} \bar{\chi}_{\dot{\alpha}}^+ \bar{D}^{\dot{\alpha}-} (D^-)^2 \mathcal{V} \\ & + \frac{1}{16} \mathcal{G}^{++} (D^-)^2 (\bar{D}^-)^2 \mathcal{V} . \end{aligned} \quad (4.24)$$

Integrating by parts and rearranging terms, we find

$$\begin{aligned} \mathcal{L}^{--} = & \frac{1}{4} F (\bar{D}^-)^2 \mathcal{V} + \frac{1}{4} \bar{F} (D^-)^2 \mathcal{V} + \frac{i}{4} \tilde{H}_{\alpha\dot{\alpha}} [D^{\alpha-}, \bar{D}^{\dot{\alpha}-}] \mathcal{V} \\ & + \mathcal{G}^{--} \square \mathcal{V} + \frac{i}{2} \mathcal{G}^{--} \partial_{\alpha\dot{\alpha}} D^{\alpha-} \bar{D}^{\dot{\alpha}-} \mathcal{V} + \frac{1}{16} \mathcal{G}^{++} (D^-)^2 (\bar{D}^-)^2 \mathcal{V} \\ & - \frac{1}{4} \chi^{\alpha+} D_\alpha^- (\bar{D}^-)^2 \mathcal{V} + \frac{1}{4} \bar{\chi}_{\dot{\alpha}}^+ \bar{D}^{\dot{\alpha}-} (D^-)^2 \mathcal{V} \\ & - i\bar{\chi}^{\dot{\beta}-} \partial_{\alpha\dot{\beta}} D^{\alpha-} \mathcal{V} - i\chi^{\alpha-} \partial_{\alpha\dot{\beta}} D^{\dot{\beta}-} \mathcal{V} . \end{aligned} \quad (4.25)$$

To simplify further, we need the identity

$$\begin{aligned} \frac{1}{16} \mathcal{G}^{++} (D^-)^2 (\bar{D}^-)^2 \mathcal{V} = & -\frac{1}{16} \mathcal{G}^{--} (D^+)^2 (\bar{D}^-)^2 \mathcal{V} + \frac{1}{16} \mathcal{G}^{++} D^{\alpha-} D_\alpha^+ (\bar{D}^-)^2 \mathcal{V} \\ & + \frac{1}{16} \mathcal{G}^{++} D^{\alpha+} D_\alpha^- (\bar{D}^-)^2 \mathcal{V} + \frac{1}{16} \mathcal{G}^{ij} D_{ij} (\bar{D}^-)^2 \mathcal{V} . \end{aligned} \quad (4.26)$$

Then using the property that \mathcal{V} is annihilated by D_α^+ , we find

$$\frac{1}{16} \mathcal{G}^{++} (D^-)^2 (\bar{D}^-)^2 \mathcal{V} = -\mathcal{G}^{--} \square \mathcal{V} - \frac{i}{2} \mathcal{G}^{--} \partial_{\alpha\dot{\alpha}} D^{\alpha-} \bar{D}^{\dot{\alpha}-} \mathcal{V} + \frac{1}{16} \mathcal{G}^{ij} D_{ij} (\bar{D}^-)^2 \mathcal{V} . \quad (4.27)$$

Similarly, we can rewrite

$$\begin{aligned} -\frac{1}{4} \chi^{\alpha+} D_\alpha^- (\bar{D}^-)^2 \mathcal{V} = & -\frac{1}{4} \chi^{\alpha-} D_\alpha^+ (\bar{D}^-)^2 \mathcal{V} - \frac{1}{4} \chi^{\alpha j} D_{\alpha j} (\bar{D}^-)^2 \mathcal{V} \\ = & i\chi^{\alpha-} \partial_{\alpha\dot{\beta}} D^{\dot{\beta}-} \mathcal{V} + \frac{1}{4} \chi_j^\alpha D_\alpha^j (\bar{D}^-)^2 \mathcal{V} . \end{aligned} \quad (4.28)$$

Together these lead to

$$\begin{aligned} \mathcal{L}^{--} = & \frac{1}{4} F (\bar{D}^-)^2 \mathcal{V} + \frac{1}{4} \bar{F} (D^-)^2 \mathcal{V} + \frac{i}{4} \tilde{H}_{\alpha\dot{\alpha}} [D^{\alpha-}, \bar{D}^{\dot{\alpha}-}] \mathcal{V} \\ & + \frac{1}{16} \mathcal{G}^{ij} D_{ij} (\bar{D}^-)^2 \mathcal{V} + \frac{1}{4} \chi_j^\alpha D_\alpha^j (\bar{D}^-)^2 \mathcal{V} + \frac{1}{4} \bar{\chi}_{\dot{\alpha}}^j \bar{D}_{\dot{\alpha}}^j (D^-)^2 \mathcal{V} . \end{aligned} \quad (4.29)$$

Applying the contour integral and taking the lowest component, $\mathcal{L} = \frac{1}{2\pi} \oint u^{i+} du_i^+ \mathcal{L}^{--}|$, leads to

$$\begin{aligned} \mathcal{L} &= \left[F\mathcal{W} + \bar{F}\bar{\mathcal{W}} - 2\tilde{H}^m v_m + \frac{1}{4} G^{ij} (D_{ij}\mathcal{W}) + \chi_j^\alpha (D_\alpha^j \mathcal{W}) + \bar{\chi}_{\dot{\alpha}}^j (\bar{D}_{\dot{\alpha}}^j \bar{\mathcal{W}}) \right] | \\ &= F\phi + \bar{F}\bar{\phi} - 2\tilde{h}^m v_m + \frac{1}{4} G^{ij} X_{ij} + \chi_j^\alpha \lambda_\alpha^j + \bar{\chi}_{\dot{\alpha}}^j \bar{\lambda}_{\dot{\alpha}}^j, \end{aligned} \quad (4.30)$$

where \mathcal{W} is given by (4.17) and we have defined

$$v_m := -\frac{i}{8} (\tilde{\sigma}_m)^{\dot{\alpha}\alpha} \oint \frac{u^{i+} du_i^+}{2\pi} [D_\alpha^-, \bar{D}_{\dot{\alpha}}^-] \mathcal{V} |. \quad (4.31)$$

It is straightforward to check that f_{mn} , defined by the flat superspace limit of eq. (3.20), is consistent with $f_{mn} = \partial_m v_n - \partial_n v_m$. Then this action may be easily compared with the flat space limit of (3.56); these differ only by an integration by parts which converts the $h \wedge v$ term into a $b \wedge f$ term.

Having constructed the globally superconformal component action, our next task is to generalize it to the locally superconformal case. In principle, this is a straightforward (if tedious) calculation making use of (4.13), but before summarizing that approach, we will briefly discuss how to generalize the equation (4.31) to the locally superconformal case.

4.4.2 Construction of the abelian vector multiplet one-form

At leading order, we expect v_m to be given (up to an exact piece which we cannot determine) by

$$v_m := -\frac{i}{8} (\tilde{\sigma}_m)^{\dot{\alpha}\alpha} \oint \frac{u^{i+} du_i^+}{2\pi} [\nabla_\alpha^-, \bar{\nabla}_{\dot{\alpha}}^-] \mathcal{V} | + \dots \quad (4.32)$$

We can deduce the missing terms by imposing the requirement of S -supersymmetry invariance of v_m . Observing that

$$S_{\beta j} [\nabla_\alpha^-, \bar{\nabla}_{\dot{\alpha}}^-] \mathcal{V} = 8\varepsilon_{\beta\alpha} u_j^- \bar{\nabla}_{\dot{\alpha}}^- \mathcal{V}, \quad (4.33)$$

we may add a gravitino term to counter this. The gravitino transforms homogeneously under S -supersymmetry,

$$S_{\beta j} \psi_{m_i}^\alpha = 0, \quad S_{\beta j} \bar{\psi}_{m_{\dot{\alpha}}}^i = -2i(\sigma_m)_{\beta\dot{\alpha}} \delta_j^i, \quad (4.34)$$

so we guess

$$v_m := -\frac{i}{8} \oint \frac{u^{i+} du_i^+}{2\pi} \left((\tilde{\sigma}_m)^{\dot{\alpha}\alpha} [\nabla_\alpha^-, \bar{\nabla}_{\dot{\alpha}}^-] \mathcal{V} - 4i\psi_m^{\alpha-} \nabla_\alpha^- \mathcal{V} + 4i\bar{\psi}_{m_{\dot{\alpha}}}^- \bar{\nabla}_{\dot{\alpha}}^- \mathcal{V} + \dots \right) |. \quad (4.35)$$

These new terms introduce additional terms since

$$S_{\beta k} \nabla_\alpha^- \mathcal{V} = -4u_k^+ \varepsilon_{\beta\alpha} D^{--} \mathcal{V}, \quad S_{\beta k} \bar{\nabla}_{\dot{\alpha}}^- \mathcal{V} = 0. \quad (4.36)$$

These can be cancelled by observing that the $SU(2)$ connection transforms homogeneously under S -supersymmetry,

$$S_{\beta k} \phi_m^i{}_j = 2\delta_k^i \psi_{m\beta j} - \delta_j^i \psi_{m\beta k} . \quad (4.37)$$

So we may choose

$$\begin{aligned} v_m &:= -\frac{i}{8} \oint \frac{u^{i+} du_i^+}{2\pi} \left((\tilde{\sigma}_m)^{\dot{\alpha}\alpha} [\nabla_\alpha^-, \bar{\nabla}_{\dot{\alpha}}^-] \mathcal{V} - 4i\psi_m^{\alpha-} \nabla_\alpha^- \mathcal{V} + 4i\bar{\psi}_{m\dot{\alpha}}^- \bar{\nabla}^{\dot{\alpha}-} \mathcal{V} + 8i\phi_m^{-+} D^{--} \mathcal{V} \right) | \\ &= \oint \frac{u^{i+} du_i^+}{2\pi} \left(-\frac{i}{8} (\tilde{\sigma}_m)^{\dot{\alpha}\alpha} [\nabla_\alpha^-, \bar{\nabla}_{\dot{\alpha}}^-] \mathcal{V} - \frac{1}{2} \psi_m^{\alpha-} \nabla_\alpha^- \mathcal{V} + \frac{1}{2} \bar{\psi}_{m\dot{\alpha}}^- \bar{\nabla}^{\dot{\alpha}-} \mathcal{V} - \phi_m^{--} \mathcal{V} \right) | , \end{aligned} \quad (4.38)$$

where we integrated D^{--} by parts under the contour integral. This expression is fully invariant under S -supersymmetry.

Having postulated an expression for v_m , we should check that it is actually sensible. Since it should generate a field strength f_{mn} that is an $SU(2)$ invariant, the expression for v_m should transform under an $SU(2)$ transformation at most by an exact form. We begin by writing

$$v_m = \oint \frac{u^{i+} du_i^+}{2\pi} \mathcal{V}_m^{--} | . \quad (4.39)$$

To avoid the inhomogeneous term from the connection piece, let's consider for the moment a global $SU(2)$ transformation under which

$$\delta \mathcal{V}_m^{--} = \lambda^{ij} J_{ij} \mathcal{V}_m^{--} = (-\lambda^{++} D^{--} + \lambda^{-+} D^0 + \lambda^{--} D^{++}) \mathcal{V}_m^{--} . \quad (4.40)$$

Under the integral, one finds

$$\delta v_m = \oint \frac{u^{i+} du_i^+}{2\pi} \lambda^{--} D^{++} \mathcal{V}_m^{--} | , \quad (4.41)$$

which is nonzero precisely because \mathcal{V}_m^{--} depends on the isotwistor u_i . We find

$$D^{++} \mathcal{V}_m^{--} | = e_m^a \nabla_a \mathcal{V} | - \frac{1}{2} \psi_m^{\alpha+} \nabla_\alpha^- \mathcal{V} | + \frac{1}{2} \bar{\psi}_{m\dot{\alpha}}^+ \bar{\nabla}^{\dot{\alpha}-} \mathcal{V} | + \phi_m^{++} D^{--} \mathcal{V} | . \quad (4.42)$$

Now observe that

$$e_m^a \nabla_a \mathcal{V} | = \partial_m \mathcal{V} | + \frac{1}{2} \psi_m^{\alpha+} \nabla_\alpha^- \mathcal{V} | - \frac{1}{2} \bar{\psi}_{m\dot{\alpha}}^+ \bar{\nabla}^{\dot{\alpha}-} \mathcal{V} | - \phi_m^{++} D^{--} \mathcal{V} | , \quad (4.43)$$

and so we see immediately that $D^{++} \mathcal{V}_m^{--} | = \partial_m \mathcal{V} |$. Thus,

$$\delta v_m = \oint \frac{u^{i+} du_i^+}{2\pi} \lambda^{--} \partial_m \mathcal{V} | . \quad (4.44)$$

If we allow $\lambda^i{}_j$ to be a local gauge transformation, the only missing piece is the inhomogeneous term from the $SU(2)$ connection. That extra piece is easily found to be

$$- \oint \frac{u^{i+} du_i^+}{2\pi} \delta \phi_m^{--} \mathcal{V} | = \oint \frac{u^{i+} du_i^+}{2\pi} \partial_m \lambda^{--} \mathcal{V} | , \quad (4.45)$$

which gives

$$\delta v_m = \partial_m \oint \frac{u^{i+} du_i^+}{2\pi} \lambda^{--} \mathcal{V} | . \quad (4.46)$$

We observe that our expression for v_m is actually *not* $SU(2)$ invariant, but it *does* generate an $SU(2)$ invariant field strength f_{mn} .

Next, we observe that v_m explicitly depends on the choice of the auxiliary isotwistor u_i (as noted above). However, f_{mn} should not, so v_m should transform into an exact form under small deformations of u_i . Let us consider a shift $u_i \rightarrow u_i + \delta u_i$ where δu_i is a constant. Due to the linear independence of v_i and u_i , it is possible to parametrize¹⁸

$$\delta u_i = v^k u_k v_i \alpha^{--} + u_i \beta , \quad \alpha^{--} := -\frac{u^i \delta u_i}{(v^k u_k)^2} , \quad \beta := \frac{v^i \delta u_i}{v^k u_k} . \quad (4.47)$$

We have chosen normalizations so that α^{--} and β , which are respectively of degree -2 and degree 0 in v^i , are both of degree zero in u_i . Now for any function $\mathcal{F}(v, u)$ which is degree zero in u_i ,

$$u_i \frac{\partial}{\partial u_i} \mathcal{F} = 0 , \quad (4.48)$$

one can easily show that

$$\delta \mathcal{F} = \alpha^{--} D^{++} \mathcal{F} , \quad (4.49)$$

under the shift (4.47). Applying this to the definition for v_m leads immediately via arguments we made above (4.44) to

$$\delta v_m = \oint \frac{u^{i+} du_i^+}{2\pi} \alpha^{--} \partial_m \mathcal{V} | = \partial_m \oint \frac{u^{i+} du_i^+}{2\pi} \alpha^{--} \mathcal{V} | , \quad (4.50)$$

since α^{--} is constant in x . As required, the expression for v_m is independent of u_i up to a gauge transformation.

Now let us verify that v_m indeed generates the correct field strength f_{mn} . Using the definition (3.19), it is straightforward to calculate

$$\begin{aligned} f_{mn} = 2 \oint \frac{u^{i+} du_i^+}{2\pi} \left[-\nabla'_{[m} \left(\frac{i}{8} (\tilde{\sigma}_n)^{\dot{\alpha}\alpha} [\nabla_{\alpha}^-, \bar{\nabla}_{\dot{\alpha}}^-] \mathcal{V} | + \frac{1}{2} \psi_n^{\alpha-} \nabla_{\alpha}^- \mathcal{V} | - \frac{1}{2} \bar{\psi}_n^{\dot{\alpha}-} \bar{\nabla}_{\dot{\alpha}}^- \mathcal{V} | \right) \right. \\ \left. - \partial_{[m} \phi_n]^{--} \mathcal{V} | + \phi_{[m}^{-j} \phi_{n]j}^- \mathcal{V} | \right] . \end{aligned} \quad (4.51)$$

In the above expression, we have used

$$\nabla'_m := \partial_m + \frac{1}{2} \omega_m^{ab} M_{ab} + i A_m Y + b_m \mathbb{D} + \phi_m^{ij} J_{ij} \quad (4.52)$$

¹⁸Here we follow very closely the approach used in [58], but with slight changes in notation.

as the derivative covariant with respect to Lorentz transformations, $SU(2) \times U(1)$ transformations, and dilatations. However, only the $SU(2)$ connection is nontrivial when acting on the term in the first line. If we extract that $SU(2)$ connection and evaluate it directly, we find

$$f_{mn} = 2 \oint \frac{u^{i+} du_i^+}{2\pi} \partial_{[m} \mathcal{V}_{n]}^{--} = 2 \partial_{[m} v_{n]} . \quad (4.53)$$

This explicit definition for v_m is especially interesting since in the application to models of one or more tensor multiplets, it is natural for the composite field strength f_{mn} which appears to be closed but not exact [55]. As an example, let us attempt to calculate v_m for the case $\mathcal{V} = \log(\mathcal{G}^{++}/i\Upsilon^+\check{\Upsilon}^+)$. In calculating v_m , there will be terms dependent on the arctic multiplet and its conjugate plus those dependent on just \mathcal{G}^{++} . We expect the dependence on the arctic multiplet to yield an exact term, so we focus only on the terms involving \mathcal{G}^{++} . It is straightforward to calculate

$$\begin{aligned} \mathcal{V}_m^{--} = & \frac{1}{G^{++}} \left(\partial_m G^{-+} - e_m^a \tilde{H}_a - \frac{1}{2}(\psi_m^j \chi_j) - \frac{1}{2}(\bar{\psi}_m^j \bar{\chi}_j) - \phi_m^{ij} G_{ij} \right) \\ & - \frac{i}{(G^{++})^2} \chi^+ \sigma_m \bar{\chi}^+ . \end{aligned} \quad (4.54)$$

A key feature of this expression is that only the first term depends on the auxiliary isotwistor u_i .

If we perform the contour integral using the techniques discussed, for example, in [53], we find the useful formulae¹⁹

$$\frac{1}{2\pi} \oint u^{i+} du_i^+ \frac{1}{G^{++}} = -\frac{1}{2} \frac{1}{G} , \quad \frac{1}{2\pi} \oint u^{i+} du_i^+ \frac{\Omega^{++}}{(G^{++})} = -\frac{1}{4} \frac{\Omega^{ij} G_{ij}}{G^3} , \quad (4.55)$$

for an arbitrary $\Omega^{++} = \Omega^{ij} u_i^+ u_j^+$. This leads to

$$\begin{aligned} v_m = & \frac{1}{G} \left(\frac{1}{2} \phi_m^{ij} G_{ij} + \frac{1}{2} e_m^a \tilde{H}_a + \frac{1}{4}(\psi_m^j \chi_j) + \frac{1}{4}(\bar{\psi}_m^j \bar{\chi}_j) \right) + \frac{i}{4G^3} (\chi^i \sigma_m \bar{\chi}^j) G_{ij} \\ & + \frac{1}{2\pi} \oint u^{i+} du_i^+ \frac{\partial_m G^{-+}}{G^{++}} . \end{aligned} \quad (4.56)$$

As expected there is a contribution to v_m which is not invariant under shifts in the auxiliary isotwistor u_i . This naturally leads to a field strength f_{mn} which is closed but not exact. However, upon taking the exterior derivative of v_m , this dependence vanishes,

$$\begin{aligned} f_{mn} = & 2 \partial_{[m} \left(\frac{1}{2G} \phi_{n]}^{ij} G_{ij} + \frac{1}{2G} e_{n]}^a \tilde{H}_a + \frac{1}{4G} (\psi_{n]}^j \chi_j) + \frac{1}{4G} (\bar{\psi}_{n]}^j \bar{\chi}_j) + \frac{i}{4G^3} (\chi^i \sigma_{n]} \bar{\chi}^j) G_{ij} \right) \\ & - \frac{1}{2\pi} \oint u^{i+} du_i^+ \frac{\partial_m G^{+j} \partial_n G_j^+}{(G^{++})^2} . \end{aligned} \quad (4.57)$$

¹⁹The contour is performed so that it encircles one of the roots of G^{++} in a counter-clockwise fashion.

The remaining contour integral can then be performed to yield

$$f_{mn} = 2\partial_{[m} \left(\frac{1}{2G} \phi_{n]}^{ij} G_{ij} + \frac{1}{2G} e_{n]}^a \tilde{H}_a + \frac{1}{4G} (\psi_n]^j \chi_j) + \frac{1}{4G} (\bar{\psi}_n]^j \bar{\chi}_j) + \frac{i}{4G^3} (\chi^i \sigma_n] \bar{\chi}^j) G_{ij} \right) + \frac{1}{4} \frac{\partial_m G^{ik} \partial_n G_k{}^j G_{ij}}{G^3} . \quad (4.58)$$

The last term is closed but cannot be written as an exact form in an $SU(2)$ -covariant way.

4.4.3 Locally superconformal action

Armed now with a reasonable expression for v_m , we may calculate the locally superconformal action. First, we present the final result

$$\begin{aligned} \mathcal{L} = & \frac{1}{4} G^{ij} X_{ij} + \left[F\phi + \chi_j^\alpha \lambda_\alpha^j + \text{c.c.} \right] - 2\tilde{h}^m v_m \\ & + \left[i(\psi_m^j \sigma^m \bar{\chi}_j) \bar{\phi} - \frac{i}{2} (\psi_m^j \sigma^m \bar{\lambda}^k) G_{jk} - (\psi_m^i \sigma^{mn} \psi_n^j) G_{ij} \bar{\phi} + \text{c.c.} \right] , \end{aligned} \quad (4.59)$$

which exactly matches (3.56), as it must, up to a total derivative. We now sketch the derivation in the remainder of this section. Readers uninterested in the details are urged to continue onward to section 5.

We begin by separating the terms in the component Lagrangian (4.13) into four groups, which we denote T_n depending on the number n of explicit gravitinos:

$$T_0 = \frac{1}{16} (\nabla^-)^2 (\bar{\nabla}^-)^2 \mathcal{L}^{++} + \frac{i}{4} \phi^{\dot{\alpha}\alpha--} [\nabla_\alpha^-, \bar{\nabla}_{\dot{\alpha}}^-] \mathcal{L}^{++} , \quad (4.60a)$$

$$T_1 = -\frac{i}{8} (\bar{\psi}_m^- \tilde{\sigma}^m)^\alpha \nabla_\alpha^- (\bar{\nabla}^-)^2 \mathcal{L}^{++} + 2(\psi_m^- \sigma^{mn})^\alpha \phi_n^{--} \nabla_\alpha^- \mathcal{L}^{++} + \text{c.c.} , \quad (4.60b)$$

$$\begin{aligned} T_2 = & -\frac{1}{4} (\psi_m^- \sigma^{mn} \psi_n^-) \mathcal{L}^{++} - \frac{1}{4} (\psi_m^- \sigma^{mn})^\alpha \bar{\psi}_m^{\dot{\alpha}-} [\nabla_\alpha^-, \bar{\nabla}_{\dot{\alpha}}^-] \mathcal{L}^{++} \\ & - \frac{3}{2} \varepsilon^{mnpq} \phi_q^{--} (\psi_m^i \sigma_n \bar{\psi}_p^j) \mathcal{L}^{++} + \text{c.c.} , \end{aligned} \quad (4.60c)$$

$$T_3 = -\frac{1}{2} \varepsilon^{mnpq} (\psi_m^- \sigma_n \bar{\psi}_p^-) \psi_q^{\alpha-} \nabla_\alpha^- \mathcal{L}^{++} + \text{c.c.} \quad (4.60d)$$

We begin with T_0 and repeat essentially the same analysis we performed in the flat case, only this time we must take curvatures into account and we cannot dispense so easily with

total derivatives. We find

$$\begin{aligned}
T_0 = & \frac{1}{4}F(\bar{\nabla}^-)^2\mathcal{V} + \frac{1}{4}\bar{F}(\nabla^-)^2\mathcal{V} + \frac{i}{4}\tilde{H}_{\alpha\dot{\alpha}}[\nabla^{\alpha-}, \bar{\nabla}^{\dot{\alpha}-}]\mathcal{V} + \frac{1}{16}\mathcal{G}^{ij}\nabla_{ij}(\bar{\nabla}^-)^2\mathcal{V} \\
& + \frac{1}{4}\chi_j^\alpha\nabla_\alpha^j(\bar{\nabla}^-)^2\mathcal{V} + \frac{1}{4}\bar{\chi}_{\dot{\alpha}}^j\bar{\nabla}_{\dot{j}}^{\dot{\alpha}}(\nabla^-)^2\mathcal{V} \\
& - \frac{1}{2}\nabla^{\beta+}W_{\beta\alpha}\mathcal{G}^{--}\nabla^{\alpha-}\mathcal{V} + \frac{1}{2}\bar{\nabla}_{\dot{\beta}}^+W^{\dot{\beta}\dot{\alpha}}\mathcal{G}^{--}\bar{\nabla}_{\dot{\alpha}}^-\mathcal{V} \\
& + \nabla^{\beta-}W_{\beta\alpha}\mathcal{G}^{-+}\nabla^{\alpha-}\mathcal{V} - \bar{\nabla}_{\dot{\beta}}^-\bar{W}^{\dot{\beta}\dot{\alpha}}\mathcal{G}^{-+}\bar{\nabla}_{\dot{\alpha}}^-\mathcal{V} \\
& + \chi^{\alpha-}\nabla^{\beta-}W_{\beta\alpha}\mathcal{V} + \bar{\chi}_{\dot{\alpha}}^-\bar{\nabla}_{\dot{\beta}}^-\bar{W}^{\dot{\beta}\dot{\alpha}}\mathcal{V} \\
& - \phi^{\dot{\alpha}\alpha--}\left(\tilde{H}_{\alpha\dot{\alpha}}\mathcal{V} - \nabla_{\alpha\dot{\alpha}}\mathcal{G}^{-+}\mathcal{V} - \frac{i}{2}\mathcal{G}^{++}\nabla_{\alpha}^-\bar{\nabla}_{\dot{\alpha}}^-\mathcal{V} + i\chi_{\alpha}^+\bar{\nabla}_{\dot{\alpha}}^-\mathcal{V} + i\bar{\chi}_{\dot{\alpha}}^+\nabla_{\alpha}^-\mathcal{V}\right) \\
& + \nabla^{\dot{\alpha}\alpha}\left(\frac{1}{2}\mathcal{G}^{--}\nabla_{\alpha\dot{\alpha}}\mathcal{V} - \frac{1}{2}\nabla_{\alpha\dot{\alpha}}\mathcal{G}^{--}\mathcal{V} - \frac{i}{2}\mathcal{G}^{-+}\nabla_{\alpha}^-\bar{\nabla}_{\dot{\alpha}}^-\mathcal{V} + i\bar{\chi}_{\dot{\alpha}}^-\nabla_{\alpha}^-\mathcal{V} + i\chi_{\alpha}^-\bar{\nabla}_{\dot{\alpha}}^-\mathcal{V}\right).
\end{aligned} \tag{4.61}$$

The first two lines match the flat space result, the next four contain conformal supergravity corrections, and the last appears to be a total derivative. This, of course, is not actually the case. The vector derivative ∇_a contains a spin-connection with torsion along with several connections – supersymmetry, SU(2), special conformal and S -supersymmetry – which do not vanish when we rewrite this as a surface term. Let us demonstrate this explicitly.

Given an action of the form $\int d^4x e \nabla_a V^a$ for some vector V^a , one can show that the expression can be rearranged into the form

$$\int d^4x \left(\nabla_m(e V^a e_a^m) - \frac{e}{2}\psi_{aj}^\beta \nabla_\beta^j V^a - \frac{e}{2}\bar{\psi}_{a\dot{\beta}}^j \bar{\nabla}_{\dot{j}}^{\dot{\beta}} V^a + e T_{mn}{}^b e_b^n V^a e_a^m \right), \tag{4.62}$$

where the torsion tensor is $T_{mn}{}^a = -i(\psi_{[mj}\sigma^a\bar{\psi}_{n]}^j)$. Moreover, the full covariant derivative ∇_m still carries connections (the special conformal and S -supersymmetry) which do not necessarily annihilate $e V^a e_a^m$.

In our case, we have a slightly more complicated situation with an auxiliary contour integration

$$\frac{1}{2\pi} \oint u^{i+} du_i^+ \int d^4x e \nabla_a V^{a--}, \tag{4.63}$$

and so we must pay attention to not only the special conformal and S -supersymmetry connections but also to the SU(2) connection within ∇_a . Taking all of these considerations into account, we find (after performing a contour integration by parts)

$$\begin{aligned}
& \frac{1}{2\pi} \oint u^{i+} du_i^+ \int d^4x e \nabla_a V^{a--} \\
& = \frac{1}{2\pi} \oint u^{i+} du_i^+ \int d^4x e \left(-\frac{1}{2}\psi_{aj}^\beta \nabla_\beta^j V^a - \frac{1}{2}\bar{\psi}_{a\dot{\beta}}^j \bar{\nabla}_{\dot{j}}^{\dot{\beta}} V^a + T_{mn}{}^b e_b^n V^a e_a^m \right. \\
& \quad \left. + \mathfrak{f}_a{}^b K_b V^{a--} + \frac{1}{2}\phi_{a\beta}^j S_j^\beta V^{a--} + \frac{1}{2}\phi_{a\dot{j}}^{\dot{\beta}} \bar{S}_{\dot{\beta}}^{\dot{j}} V^{a--} + \phi_a{}^{--} D^{++} V^{a--} \right).
\end{aligned} \tag{4.64}$$

We emphasize that the operator D^{++} *cannot* be integrated by parts under the contour integral, since it involves a derivative with respect to the auxiliary isotwistor u_i which is not a variable of integration.

If we now integrate by parts the vector derivative in T_0 , we find

$$\begin{aligned}
T_0 = & \frac{1}{4}F(\bar{\nabla}^-)^2\mathcal{V} + \frac{1}{4}\bar{F}(\nabla^-)^2\mathcal{V} + \frac{1}{16}\mathcal{G}^{ij}\nabla_{ij}(\bar{\nabla}^-)^2\mathcal{V} \\
& + \frac{1}{4}\chi_j^\alpha\nabla_\alpha^j(\bar{\nabla}^-)^2\mathcal{V} + \frac{1}{4}\bar{\chi}_\alpha^j\bar{\nabla}_j^\alpha(\nabla^-)^2\mathcal{V} + \frac{i}{4}\tilde{H}_{\alpha\dot{\alpha}}[\nabla^{\alpha-}, \bar{\nabla}^{\dot{\alpha}-}]\mathcal{V} - \tilde{H}_{\alpha\dot{\alpha}}\phi^{\dot{\alpha}\alpha--}\mathcal{V} \\
& + \left(\frac{1}{2}\Psi^{\dot{\beta}\alpha}_{\beta\dot{\beta}}\mathcal{G}^{-+}\nabla_\alpha^-\mathcal{V} - \frac{1}{4}\Psi^{\dot{\beta}\alpha}_{\beta\dot{\beta}}\mathcal{G}^{--}\nabla_\alpha^-\mathcal{V} - \frac{1}{2}\Psi^{\dot{\beta}\alpha}_{\beta\dot{\beta}}\mathcal{G}^{-+}\chi_\alpha^-\mathcal{V} \right. \\
& \quad \left. - \frac{i}{4}\psi_{\alpha\dot{\beta}}^{\alpha+}\bar{W}^{\dot{\beta}\dot{\alpha}}\mathcal{G}^{--}\bar{\nabla}_{\dot{\alpha}}^-\mathcal{V} + \frac{i}{2}\psi_{\alpha\dot{\beta}}^{\alpha-}\bar{W}^{\dot{\beta}\dot{\alpha}}\mathcal{G}^{-+}\bar{\nabla}_{\dot{\alpha}}^-\mathcal{V} + \frac{i}{2}\psi_{\alpha\dot{\beta}}^{\alpha-}\bar{W}^{\dot{\beta}\dot{\alpha}}\bar{\chi}_{\dot{\alpha}}^-\mathcal{V} + \text{c.c.} \right) \\
& - \left(\frac{1}{2}\psi^{\dot{\alpha}\alpha\beta}_j\nabla_\beta^j + \frac{1}{2}\bar{\psi}^{\dot{\alpha}\alpha j}_\beta\bar{\nabla}_j^{\dot{\beta}} \right) \Omega_{\alpha\dot{\alpha}}^{--} - 2T_{mn}{}^b e_b{}^n e_a{}^m \Omega^{a--} .
\end{aligned} \tag{4.65}$$

It is easy to see that this result matches (4.59) to zeroth order in the gravitinos.

If we next include the contribution of T_1 , perform another set of integrations by parts, and rewrite all expressions involving $[\nabla_\alpha^-, \bar{\nabla}_{\dot{\alpha}}^-]\mathcal{V}$ to instead involve $\mathcal{V}_{\alpha\dot{\alpha}}^{--}$, we find

$$\begin{aligned}
T_0 + T_1 = & \frac{1}{4}\mathcal{G}^{ij}\nabla_{ij}\mathcal{W}^{--} + \left[F\mathcal{W}^{--} + \chi_j^\alpha\nabla_\alpha^j\mathcal{W}^{--} + \text{c.c.} \right] \\
& + \tilde{H}^{\dot{\alpha}\alpha}\mathcal{V}_{\alpha\dot{\alpha}}^{--} - \left[2(\psi_m{}^j\sigma^{mn}\chi_j)\mathcal{V}_n^{--} + \text{c.c.} \right] \\
& + \left[i\psi_{\alpha\dot{\alpha}}^{\alpha j}\bar{\chi}_j^{\dot{\alpha}}\bar{\mathcal{W}}^{--} - \frac{i}{2}\psi^{\dot{\alpha}\alpha j}_\alpha\mathcal{G}_{jk}\bar{\nabla}_{\dot{\alpha}}^k\bar{\mathcal{W}}^{--} + \text{c.c.} \right] \\
& + R_1 ,
\end{aligned} \tag{4.66}$$

where we have introduced $\mathcal{W}^{--} := \frac{1}{4}(\bar{\nabla}^-)^2\mathcal{V}$ as convenient shorthand and collected all terms involving two or more gravitinos into the remainder term R_1

$$\begin{aligned}
R_1 = & \frac{1}{4}\psi^{\dot{\alpha}\alpha\beta j}\chi_{\beta j}(\psi_{\alpha\dot{\alpha}}^{\gamma-}\nabla_\gamma^-\mathcal{V} - \bar{\psi}_{\alpha\dot{\alpha}}^{\dot{\gamma}-}\bar{\nabla}^{\dot{\gamma}-}\mathcal{V}) \\
& - \frac{1}{2}\psi^{\dot{\alpha}\alpha\beta j}\chi_{\alpha j}(\psi_{\beta\dot{\alpha}}^{\gamma-}\nabla_\gamma^-\mathcal{V} - \bar{\psi}_{\beta\dot{\alpha}}^{\dot{\gamma}-}\bar{\nabla}^{\dot{\gamma}-}\mathcal{V}) \\
& + \frac{1}{2}\left[(\psi_m{}^j\sigma^{mn})^\alpha\psi_{n\dot{\gamma}}^{\gamma}\nabla_\gamma^k + (\psi_m{}^j\sigma^{mn})^\alpha\bar{\psi}_{n\dot{\gamma}}^k\bar{\nabla}_{\dot{\gamma}}^{\dot{\gamma}} \right] \Lambda_{\alpha j}^{--} \\
& + \frac{1}{2}\varepsilon^{mnpq}(\bar{\Psi}_{mn}{}^j\bar{\sigma}_p\psi_q^i)\left(\frac{1}{2}u_i^+u_j^-\mathcal{G}^{--} + \frac{1}{2}u_i^-u_j^+\mathcal{G}^{--} - 2u_i^-u_j^-\mathcal{G}^{--} \right)\mathcal{V} \\
& + i\varepsilon^{mnpq}(\psi_m^+\sigma_{na})^\alpha T_{pq}{}^a(\mathcal{G}^{-+}\nabla_\alpha^-\mathcal{V} - \chi_\alpha^-\mathcal{V}) - \frac{i}{2}\varepsilon^{mnpq}(\psi_m^+\sigma_{na})^\alpha T_{pq}{}^a(\mathcal{G}^{--}\nabla_\alpha^-\mathcal{V}) \\
& + \text{c.c.} - 2T_{nm}{}^b e_b{}^m e_a{}^n \Omega^{a--} .
\end{aligned} \tag{4.67}$$

We have defined

$$\Lambda_{\alpha j}^{--} = \frac{1}{2}u_j^-\chi_\alpha^-\mathcal{V} + \frac{1}{4}u_j^+\mathcal{G}^{--}\nabla_\alpha^-\mathcal{V} - \frac{1}{2}u_j^-\mathcal{G}^{-+}\nabla_\alpha^-\mathcal{V} , \tag{4.68}$$

to keep the expression for R_1 somewhat simpler. One can easily check that the above expression is correct to first order in gravitinos. The term R_1 involves only two gravitinos or more and should be cancelled when additional contributions are included.

Now let us include the two gravitino term T_2 . Performing another set of integrations by parts leads to

$$\begin{aligned}
T_0 + T_1 + T_2 = & \frac{1}{4} \mathcal{G}^{ij} \nabla_{ij} \mathcal{W}^{--} + \left[F \mathcal{W}^{--} + \chi_j^\alpha \nabla_\alpha^j \mathcal{W}^{--} + \text{c.c.} \right] \\
& - \left[2\tilde{H}^q + 2(\psi_m^j \sigma^{mq} \chi_j + \text{c.c.}) - \varepsilon^{mnpq} (\psi_m^i \sigma_n \bar{\psi}_p^j) \mathcal{G}_{ij} \right] \mathcal{V}_q^{--} \\
& + \left[i(\psi_m^j \sigma^m \bar{\chi}_j) \bar{\mathcal{W}}^{--} - \frac{i}{2} (\psi_m^j \sigma^m)_{\dot{\alpha}} \mathcal{G}_{jk} \bar{\nabla}^{\dot{\alpha}k} \bar{\mathcal{W}}^{--} + \text{c.c.} \right] \\
& - \left[(\psi_m^i \sigma^{mn} \psi_n^j) \mathcal{G}_{ij} \bar{\mathcal{W}}^{--} + \text{c.c.} \right] \\
& + R_2 ,
\end{aligned} \tag{4.69}$$

where the remainder R_2 is a three-gravitino term:

$$\begin{aligned}
R_2 = & \frac{1}{2} \varepsilon^{mnpq} (\psi_m^i \sigma_n \bar{\psi}_p^j) \mathcal{G}_{ij} \psi_m^{\alpha-} \nabla_\alpha^- \mathcal{V} - \frac{1}{2} \varepsilon^{mnpq} (\psi_m^i \sigma_n \bar{\psi}_p^j) \psi_{qk}^\alpha \nabla_\alpha^k \Lambda_{ij}^{--} \\
& - i \varepsilon^{mnpq} (\psi_m^i \sigma_{na})^\alpha T_{pq}^a \left(\frac{1}{2} u_i^+ \mathcal{G}^{--} \nabla_\alpha^- \mathcal{V} - u_i^- \mathcal{G}^{--} \nabla_\alpha^- \mathcal{V} + u_i^- \chi_\alpha^- \mathcal{V} \right) + \text{c.c.}
\end{aligned} \tag{4.70}$$

Finally, we expand out T_3 and add it to R_2 . This is a straightforward exercise of performing $\text{SU}(2)$ algebra with the harmonics, and one finds that $R_2 + T_3 = 0$. So the full component Lagrangian is given by

$$\begin{aligned}
\mathcal{L}^{--} = & \frac{1}{4} \mathcal{G}^{ij} \nabla_{ij} \mathcal{W}^{--} + \left[F \mathcal{W}^{--} + \chi_j^\alpha \nabla_\alpha^j \mathcal{W}^{--} + \text{c.c.} \right] \\
& - \left[2\tilde{H}^q + 2(\psi_m^j \sigma^{mq} \chi_j + \text{c.c.}) - \varepsilon^{mnpq} (\psi_m^i \sigma_n \bar{\psi}_p^j) \mathcal{G}_{ij} \right] \mathcal{V}_q^{--} \\
& + \left[i(\psi_m^j \sigma^m \bar{\chi}_j) \bar{\mathcal{W}}^{--} - \frac{i}{2} (\psi_m^j \sigma^m)_{\dot{\alpha}} \mathcal{G}_{jk} \bar{\nabla}^{\dot{\alpha}k} \bar{\mathcal{W}}^{--} + \text{c.c.} \right] \\
& - \left[(\psi_m^i \sigma^{mn} \psi_n^j) \mathcal{G}_{ij} \bar{\mathcal{W}}^{--} + \text{c.c.} \right] .
\end{aligned} \tag{4.71}$$

All terms have been rearranged so that the contour integrals can be performed. Doing so, and taking the component projection leads to

$$\begin{aligned}
\mathcal{L} = & \frac{1}{4} G^{ij} X_{ij} + \left[F \phi + \chi_j^\alpha \Sigma_\alpha^j + \text{c.c.} \right] - 2\tilde{h}^m v_m \\
& + \left[i(\psi_m^j \sigma^m \bar{\chi}_j) \bar{\phi} - \frac{i}{2} (\psi_m^j \sigma^m)_{\dot{\alpha}} G_{jk} \bar{\Sigma}^{\dot{\alpha}k} - (\psi_m^i \sigma^{mn} \psi_n^j) G_{ij} \bar{\phi} + \text{c.c.} \right] ,
\end{aligned} \tag{4.72}$$

where we have identified the term multiplying v_m using eq. (3.52)

After a laborious calculation, we have indeed recovered (3.56). Of course, we *had* to, since a straightforward superspace argument guarantees that the projective superspace action constructed from the Lagrangian $\mathcal{L}^{++} = \mathcal{G}^{++} \mathcal{V}$ must match the chiral superspace action constructed from $\mathcal{L}_c = \Psi \mathcal{W}$. What is important about this calculation is a demonstration that we can explicitly evaluate projective superspace actions in conformal supergravity *without* first converting them to $\mathcal{N} = 1$ language or by first converting them to a chiral (or some other) Lagrangian.

5 Lifting superspace results from components

In the previous sections, we have emphasized the use of superspace for the construction of supersymmetric actions and the derivation of supersymmetry transformation rules. However, this process can also be reversed. Given a consistent set of component supersymmetry transformation rules for some multiplet, there should be a superfield formulation obeying some set of constraints which corresponds to that multiplet. In principle, the supersymmetry transformations may be used to identify the constraints on the superfield. In this section we will illustrate this point with the example of the vector-tensor multiplet constructed in [27].

The vector-tensor (VT) multiplet [15] (see also the review [43]) has received considerable attention triggered by the discovery that it describes the dilaton-axion complex in $\mathcal{N} = 2$ supersymmetric vacua of heterotic string theory [16]. The VT multiplet can be obtained from an abelian $\mathcal{N} = 2$ vector multiplet by eliminating the auxiliary field and dualizing one of the two physical fields in the vector multiplet to a gauge two-form. The resulting multiplet (once an auxiliary field is restored) is off-shell in the presence of a central charge.

A number of papers have addressed different variants of the VT multiplet using superspace techniques.²⁰ One approach is to look for consistent deformations of the superfield constraints and has been pursued by a number of authors [17–21] in flat superspace. A second line of attack is to construct the superform geometry associated with the VT multiplet by identifying its one-form and/or two-form as component projections of superspace forms [22–24]. A third method is simply to work out its component structure directly. This approach was exhaustively applied using the superconformal tensor calculus by Claus *et al.* [25–27], where it was shown that there were essentially two distinct types of VT multiplet in supergravity with a gauged central charge: the linear VT multiplet and the nonlinear VT multiplet.²¹

Recently, in [28] a superspace formulation for these two cases of the VT multiplet has been found in supergravity. However, it remains unclear how to compare these results to those given in [27]. Our goal in this section is to recast the results of [27] in a superspace form and to compare them to the results of [28].

5.1 Vector-tensor multiplet in supergravity

The vector-tensor multiplet consists of five component fields: a real scalar ℓ , a Weyl fermion λ_α^i , a real scalar U , a gauge one-form v_m , and a gauge two-form b_{mn} . The real scalar U can alternatively be understood as the action of the central charge on the real scalar ℓ . In the language of Claus *et al.*, $U = \ell^{(z)}$ [25–27].

The most general vector-tensor multiplet considered in [27] was coupled not only to the central charge vector multiplet \mathcal{Z} but also to a number of additional vector multiplets

²⁰The recent papers [31, 32] use Free Differential Algebra to formulate a general coupling of VT multiplets to $\mathcal{N} = 2$ supergravity.

²¹The linear VT multiplet must be coupled to (at least) one additional vector multiplet for consistency. Additional Chern-Simons-type couplings are also allowed in both cases.

\mathcal{W}^A with $A = 2, \dots, n$. These couplings were parametrized by a set of real numbers η_{11} , η_{1A} , and η_{AB} . The indices 1 and A label component one-forms, with the first one-form associated with the vector-tensor multiplet L and the rest associated with \mathcal{W}^A . The one-form associated with the central charge multiplet was denoted 0. Below we have attempted to match as much as possible the notation used in these papers, except for the lift to superspace.²² In this way, we will derive constraints for the vector-tensor multiplet in curved superspace.

To lift the vector-tensor multiplet of [27] to superspace, we must construct a superfield L which possesses ℓ as its lowest component. The superfield L should share the same properties as its lowest component: this means L should be real, it should have vanishing dilatation weight, and it should be conformally primary.

Our first step is a simple one: we analyze $\delta\ell$. Since $\ell = L|$, we know that

$$\delta\ell = (\xi_i^\alpha \nabla_\alpha^i L + \bar{\xi}_\alpha^i \bar{\nabla}_i^\alpha L)| = \xi_i^\alpha \lambda_\alpha^i + \bar{\xi}_\alpha^i \bar{\lambda}_i^\alpha, \quad (5.1)$$

where the second equality follows from the component approach. Hence we deduce the obvious definitions

$$\lambda_\alpha^i := \nabla_\alpha^i L|, \quad \bar{\lambda}_i^\alpha := \bar{\nabla}_i^\alpha L|. \quad (5.2)$$

The next step is nontrivial. Beginning with the relation

$$\delta\lambda_\alpha^i = (\delta\nabla_\alpha^i L)| = (\xi_j^\beta \nabla_\beta^j \nabla_\alpha^i L + \bar{\xi}_{\dot{\beta}}^j \bar{\nabla}_j^{\dot{\beta}} \nabla_\alpha^i L)|, \quad (5.3)$$

we rearrange the right-hand side into irreducible representations of $SU(2)$ and the Lorentz group. We use

$$\begin{aligned} \nabla_\beta^j \nabla_\alpha^i L &= \frac{1}{2} \{ \nabla_\beta^j, \nabla_\alpha^i \} L + \frac{1}{2} [\nabla_\beta^j, \nabla_\alpha^i] L \\ &= -\varepsilon_{\beta\alpha} \varepsilon^{ji} \bar{\mathcal{Z}} \Delta L + \frac{1}{2} \varepsilon_{\beta\alpha} \nabla^{ji} L - \frac{1}{2} \varepsilon^{ji} \nabla_{\beta\alpha} L \end{aligned} \quad (5.4)$$

and

$$\begin{aligned} \bar{\nabla}_j^{\dot{\beta}} \nabla_\alpha^i L &= \frac{1}{2} \{ \bar{\nabla}_j^{\dot{\beta}}, \nabla_\alpha^i \} L + \frac{1}{2} [\bar{\nabla}_j^{\dot{\beta}}, \nabla_\alpha^i] L \\ &= -i \delta_j^i \nabla_\alpha^{\dot{\beta}} L - \frac{1}{2} [\nabla_\alpha^i, \bar{\nabla}_j^{\dot{\beta}}] L \\ &= -i \delta_j^i \nabla_\alpha^{\dot{\beta}} L - \frac{1}{4} \delta_j^i [\nabla_\alpha^k, \bar{\nabla}_k^{\dot{\beta}}] L - \frac{1}{2} \varepsilon_{jk} [\nabla_\alpha^{(i}, \bar{\nabla}^{\dot{\beta}k)}] L. \end{aligned} \quad (5.5)$$

This leads to

$$\begin{aligned} \delta\nabla_\alpha^i L &= -\xi_\alpha^i \bar{\mathcal{Z}} \Delta L - \frac{1}{2} \xi_{j\alpha} \nabla^{ji} L + \frac{1}{2} \xi^{i\beta} \nabla_{\beta\alpha} L \\ &\quad - i \bar{\xi}_{\dot{\beta}}^i \nabla_\alpha^{\dot{\beta}} L - \frac{1}{4} \bar{\xi}_{\dot{\beta}}^i [\nabla_\alpha^k, \bar{\nabla}_k^{\dot{\beta}}] L + \frac{1}{2} \bar{\xi}_{k\dot{\beta}} [\nabla_\alpha^{(i}, \bar{\nabla}^{\dot{\beta}k)}] L. \end{aligned} \quad (5.6)$$

²²Claus *et al.* denoted X as the lowest component of the central charge vector multiplet, which we denote by \mathcal{Z} . In addition, we use a different normalization convention for the vector multiplet. However, since in the couplings to the vector-tensor multiplet all vector multiplets appear in ratios, this normalization will not matter.

Taking the component projection and comparing with the expression for $\delta\lambda_\alpha^i$ given in eq. (3.10) of [27], we can immediately conclude a number of relations by comparing the coefficients of ξ and $\bar{\xi}$. Only two of them will be of interest to us here, since they define the constraints on the vector-tensor multiplet. These consist of:

$$\nabla_\alpha^{(i} \bar{\nabla}^{j)} L = -\bar{\nabla}_{\dot{\beta}}^{(j} \nabla_\alpha^{i)} L = 0 \quad (5.7)$$

and

$$\nabla^{ij} L = -\frac{2}{\mathcal{Z}} \nabla^{\alpha(i} L \nabla_\alpha^{j)} \mathcal{Z} + \frac{1}{\mathcal{Z}} \frac{1}{2\eta_{11} L - \text{Re } g} \Gamma^{ij}, \quad (5.8a)$$

$$\begin{aligned} \Gamma^{ij} := & -\eta_{11} L^2 \nabla^{ij} \mathcal{Z} - 2\eta_{11} \mathcal{Z} \nabla^{\alpha(i} L \nabla_\alpha^{j)} L + 2\eta_{11} \bar{\mathcal{Z}} \bar{\nabla}_{\dot{\alpha}}^{(i} L \bar{\nabla}^{\dot{\alpha}j)} L \\ & - \frac{1}{2} L \bar{\mathcal{Z}} \frac{\partial \bar{g}}{\partial \bar{\mathcal{W}}^A} \bar{\nabla}^{ij} \bar{\mathcal{W}}^A - \frac{1}{2} L \bar{\mathcal{Z}} \frac{\partial \bar{g}}{\partial \bar{\mathcal{Z}}} \bar{\nabla}^{ij} \bar{\mathcal{Z}} + \mathcal{Z} \nabla^{\alpha(i} g \nabla_\alpha^{j)} L - \bar{\mathcal{Z}} \bar{\nabla}_{\dot{\alpha}}^{(i} \bar{g} \bar{\nabla}^{\dot{\alpha}j)} L \\ & + i \nabla^{ij} (\mathcal{Z} b) + i \bar{\nabla}^{ij} (\bar{\mathcal{Z}} \bar{b}). \end{aligned} \quad (5.8b)$$

The following combinations of fields have been used:

$$g := i\eta_{1A} \frac{\mathcal{W}^A}{\mathcal{Z}}, \quad b := -\frac{i}{4} \eta_{AB} \frac{\mathcal{W}^A \mathcal{W}^B}{\mathcal{Z}^2}. \quad (5.9)$$

It was observed in [27] that the constraints (and the action) are unchanged by certain constant shifts in the parameters g and b . These shifts turn out to coincide with constant shifts in the real part of $\mathcal{W}^A/\mathcal{Z}$.

The two equations (5.7) and (5.8) constitute the lift of the VT multiplet to superspace. Just as at the component level [25–27], different superspace formulations of the VT multiplet can be recovered by different choices of the coefficients η : the nonlinear VT multiplet ($\eta_{1A} = 0$) and the linear VT multiplet ($\eta_{11} = 0$), both with ($\eta_{AB} \neq 0$) and without ($\eta_{AB} = 0$) Chern-Simons terms.

5.2 Reinterpreting the constraints

Due to the importance of these constraints in describing the multiplet, we would like to have a better understanding of them. It turns out that they have an elegant physical interpretation from dimensional reduction of a 5D theory in the Minkowski limit [64]. Before giving the interpretation, let us perform a number of relabellings which will allow us to dramatically simplify (5.8).

First, let us take the quantity $\mathcal{W}^A/\mathcal{Z}$ and consider its imaginary part

$$Y^A := \frac{1}{2i} \left(\frac{\mathcal{W}^A}{\mathcal{Z}} - \frac{\bar{\mathcal{W}}^A}{\bar{\mathcal{Z}}} \right). \quad (5.10)$$

It turns out the constraint (5.8) can be rewritten entirely in terms of the quantity Y^A :

$$\nabla^{ij} L = -\frac{2}{\mathcal{Z}} \nabla^{\alpha(i} L \nabla_\alpha^{j)} \mathcal{Z} + \frac{1}{\mathcal{Z}} \frac{1}{2\eta_{11} L + \eta_{1A} Y^A} \Gamma^{ij} \quad (5.11)$$

with

$$\begin{aligned}
\Gamma^{ij} = & -\eta_{11} L^2 \nabla^{ij} \mathcal{Z} - 2\eta_{11} \mathcal{Z} \nabla^\alpha ({}^i L \nabla_\alpha^j) L + 2\eta_{11} \bar{\mathcal{Z}} \bar{\nabla}_{\dot{\alpha}} ({}^i L \bar{\nabla}^{\dot{\alpha}j}) L \\
& - \frac{1}{2} L \bar{\mathcal{Z}} \eta_{1A} Y^A \nabla^{ij} \mathcal{Z} - \frac{1}{2} L \mathcal{Z} \eta_{1A} \nabla^{ij} Y^A + \frac{1}{2} L \bar{\mathcal{Z}} \eta_{1A} \bar{\nabla}^{ij} Y^A \\
& - L \eta_{1A} \nabla^\alpha ({}^i \mathcal{Z} \nabla_\alpha^j) Y^A + L \eta_{1A} \bar{\nabla}_{\dot{\alpha}} ({}^i \bar{\mathcal{Z}} \bar{\nabla}^{\dot{\alpha}j}) Y^A \\
& - 2\mathcal{Z} \eta_{1A} \nabla^\alpha ({}^i L \nabla_\alpha^j) Y^A + 2\bar{\mathcal{Z}} \eta_{1A} \bar{\nabla}_{\dot{\alpha}} ({}^i L \bar{\nabla}^{\dot{\alpha}j}) Y^A \\
& - \mathcal{Z} \eta_{AB} Y^A \nabla^{ij} Y^B + \bar{\mathcal{Z}} \eta_{AB} Y^A \bar{\nabla}^{ij} Y^B \\
& - 2\eta_{AB} Y^A \nabla^\alpha ({}^i \mathcal{Z} \nabla_\alpha^j) Y^B + 2\eta_{AB} Y^A \bar{\nabla}_{\dot{\alpha}} ({}^i \bar{\mathcal{Z}} \bar{\nabla}^{\dot{\alpha}j}) Y^B \\
& - 2\mathcal{Z} \eta_{AB} \nabla^\alpha ({}^i Y^A \nabla_\alpha^j) Y^B + 2\bar{\mathcal{Z}} \eta_{AB} \bar{\nabla}_{\dot{\alpha}} ({}^i Y^A \bar{\nabla}^{\dot{\alpha}j}) Y^B .
\end{aligned} \tag{5.12}$$

A further dramatic simplification occurs if we consider the real part of this constraint. One finds the equation

$$0 = \mathcal{Z} \nabla^{ij} L + 2 \nabla^\alpha ({}^i L \nabla_\alpha^j) \mathcal{Z} + \frac{1}{2} L \nabla^{ij} \mathcal{Z} + \text{c.c.} , \tag{5.13}$$

which is *completely linear* in L . Together with (5.7), we have two linear constraints in L ; the imaginary part of (5.8) gives an additional *nonlinear* constraint.

Before discussing the nonlinear constraint, we should make one further observation. The superfield Y^A was constructed by taking the imaginary part of the combination $\mathcal{W}^A/\mathcal{Z}$. Since \mathcal{W}^A and \mathcal{Z} are both vector multiplets, their respective Bianchi identities should imply some sort of constraint on Y^A . Indeed, it is easy to check that

$$0 = \mathcal{Z} \nabla^{ij} Y^A + 2 \nabla^\alpha ({}^i Y^A \nabla_\alpha^j) \mathcal{Z} + \frac{1}{2} Y^A \nabla^{ij} \mathcal{Z} + \text{c.c.} , \tag{5.14}$$

which is exactly the same form as (5.13)! Furthermore, one can easily check that

$$\nabla_\alpha ({}^i \bar{\nabla}^j) Y^A = -\bar{\nabla}_{\dot{\beta}} ({}^j \nabla_\alpha^i) Y^A = 0 , \tag{5.15}$$

which is the same form as (5.7). This suggests that we ought to consider the superfield L as Y^1 and consider the set of real superfields $Y^I = (L, Y^A)$ obeying the linear constraints (5.14) and (5.15). It must be emphasized that Claus *et al.* already observed at the component level that it is profitable to interpret the one-forms within L and \mathcal{W}^A on the same footing. Our observation lifts their consideration to superspace.

Finally, let us take the imaginary part of the constraint (5.8). The equation one finds is equivalent to

$$0 = \eta_{IJ} G^{IJij} , \tag{5.16}$$

where the quantity G^{IJij} is defined by

$$\begin{aligned}
G^{IJij} := & i \mathcal{Z} \nabla^\alpha ({}^i Y^I \nabla_\alpha^j) Y^J + \frac{i}{4} \mathcal{Z} Y^I \nabla^{ij} Y^J + \frac{i}{4} \mathcal{Z} Y^J \nabla^{ij} Y^I \\
& + \frac{i}{2} Y^I \nabla^\alpha ({}^i Y^J \nabla_\alpha^j) \mathcal{Z} + \frac{i}{2} Y^J \nabla^\alpha ({}^i Y^I \nabla_\alpha^j) \mathcal{Z} + \text{c.c.}
\end{aligned} \tag{5.17}$$

and the numeric coefficients η_{IJ} are given by

$$\eta_{IJ} = \begin{pmatrix} \eta_{11} & \eta_{1A} \\ 0 & \eta_{AB} \end{pmatrix}, \quad (5.18)$$

with η_{A1} defined to vanish. The equation (5.16) is bilinear in $Y^I = (L, Y^A)$. The imaginary part of the constraint (5.8) arises by solving (5.16) for $\nabla^{ij}L - \bar{\nabla}^{ij}L$, with non-linearities arising when $\eta_{11} \neq 0$.

There is a simple 5D interpretation of these results, at least in the Minkowski limit, due to Kuzenko and Linch [64].²³ The nonlinear vector-tensor multiplet in 4D Minkowski space naturally arises from dimensional reduction of a 5D Chern-Simons theory. Within 5D superspace, the real 5D vector multiplet Y^I is governed by two Bianchi identities, which can be written as (5.14) and (5.15). If one constructs a 5D Chern-Simons action in superspace, its *equation of motion* matches (5.16) in the Minkowski limit. In dimensionally reducing this action to 4D, this equation of motion is reinterpreted as a 4D constraint on $Y^1 = L$.

A second observation we would like to make is how the above reformulation clarifies an observation made in [26]. There it was noted that for general VT multiplets with non-linearities ($\eta_{11} \neq 0$), it was possible to make certain redefinitions of the components of the VT multiplet so that η_{1A} vanishes. In the reformulation given above, this property is manifest. First, we observe that since both the superfields L and Y^A obey the same linear constraints – (5.7) and (5.13) for L and (5.15) and (5.14) for Y^A – it is possible to *redefine* L by adding to it a certain linear combination of Y^A ,

$$L = L' + c_A Y^A, \quad (5.19)$$

where c_A is some real constant. Due to the bilinearity of the remaining constraint (5.16), one can easily show that this transformation can be countered by a redefinition of the parameters η_{IJ}

$$\eta_{11} = \eta'_{11}, \quad \eta_{1A} = \eta'_{1A} - 2\eta'_{11}c_A, \quad \eta_{AB} = \eta'_{AB} - \eta'_{1(A}c_{B)} + \eta'_{11}c_Ac_B \quad (5.20)$$

so that the form of the constraint remains unchanged. In particular, the choice $c_A = -\eta_{1A}/2\eta_{11}$ will set $\eta'_{1A} = 0$, as noted in [26]. This property implies that for the nonlinear VT multiplet, the most general case can always be mapped to the case $\eta_{1A} = 0$ and vice-versa by a superfield redefinition.

There is a third, technical observation which we wish to make about the constraints. In a series of papers, formulations of the VT multiplet have been worked out in the case of conventional $\mathcal{N} = 2$ superspace [21–24] and $\mathcal{N} = 2$ harmonic superspace [17–20] by making use of consistency conditions. A recent paper [28], using a similar approach has found a superspace formulation in terms of superfields for the cases of the linear and non-linear VT multiplets within $SU(2)$ superspace. One of the consistency conditions has an

²³Kuzenko and Linch [64] considered the nonlinear case with $\eta_{1A} = \eta_{AB} = 0$, but the generalization is completely straightforward.

interesting relationship to the constraints when written in projective superspace. Since L is independent of the isotwistors, we have

$$0 = D^{--}L = D^{++}L = D^0L . \quad (5.21)$$

Recalling that $\nabla_\alpha^\pm := u_i^\pm \nabla_\alpha^i$ and $\bar{\nabla}_{\dot{\alpha}}^\pm := u_i^\pm \bar{\nabla}_{\dot{\alpha}}^i$ in the language of section 4, we may apply successive gauged covariant derivatives to the first condition of (5.21) to derive a consistency condition:

$$\begin{aligned} 0 &= \nabla^{\alpha+} \nabla_\alpha^+ \bar{\nabla}_{\dot{\alpha}}^+ \bar{\nabla}^{\dot{\alpha}+} D^{--}L \\ &= D^{--} \nabla^{\alpha+} \nabla_\alpha^+ \bar{\nabla}_{\dot{\alpha}}^+ \bar{\nabla}^{\dot{\alpha}+} L + 8i \nabla^{\alpha\dot{\alpha}} \nabla_\alpha^+ \bar{\nabla}_{\dot{\alpha}}^+ L \\ &\quad - 2 \nabla^{\alpha-} \nabla_\alpha^+ \bar{\nabla}_{\dot{\alpha}}^+ \bar{\nabla}^{\dot{\alpha}+} L - 2 \bar{\nabla}_{\dot{\alpha}}^- \bar{\nabla}^{\dot{\alpha}+} \nabla^{\alpha+} \nabla_\alpha^+ L \\ &\quad - 4\Delta \left((\nabla^+)^2 (\mathcal{Z}L) + (\bar{\nabla}^+)^2 (\bar{\mathcal{Z}}L) - \frac{1}{2} L (\nabla^+)^2 \mathcal{Z} - \frac{1}{2} L (\bar{\nabla}^+)^2 \bar{\mathcal{Z}} \right) . \end{aligned} \quad (5.22)$$

This places restrictions on the possible constraints for L . For instance, if we impose the constraint (5.7), we have

$$0 = \Delta \left((\nabla^+)^2 (\mathcal{Z}L) + (\bar{\nabla}^+)^2 (\bar{\mathcal{Z}}L) - L (\nabla^+)^2 \mathcal{Z} \right) . \quad (5.23)$$

This is guaranteed by the (slightly stronger) constraint

$$0 = \nabla^{ij} (\mathcal{Z}L) + \bar{\nabla}^{ij} (\bar{\mathcal{Z}}L) - L \nabla^{ij} \mathcal{Z} , \quad (5.24)$$

which is equivalent to (5.13). It is remarkable that the second linear constraint on L may be motivated from the first solely on the basis of consistency.

5.3 Action formulation for the vector-tensor multiplet

The action principle for the VT multiplet requires a composite real linear multiplet L^{ij} ,

$$\nabla_\alpha^{(i} L^{jk)} = \bar{\nabla}_{\dot{\alpha}}^{(i} L^{jk)} = 0 , \quad (5.25)$$

which transforms under the central charge. If this linear multiplet is coupled to conformal supergravity using the vector multiplet \mathcal{Z} that gauges the central charge, an invariant action can be constructed at the component level (see *e.g.* the discussion in [26, 27] or the original literature [1, 3, 42]). This action works even if the linear multiplet is invariant under the central charge, and coincides exactly in that case with (3.56) and (4.59), which we derived directly from superspace. It would be useful to have a direct projective superspace formulation of this action for non-vanishing central charge, but as yet no such formulation exists.²⁴

We may nevertheless consider the lift to superspace of the linear multiplet L^{ij} constructed by Claus *et al.* [27] and look for simplifications inspired by the observations made

²⁴In harmonic superspace, one can construct the analogue to (4.20). See the discussion in [20] for the globally supersymmetric case and [65] for the local case.

in the previous subsection. The linear multiplet L^{ij} was based on an ansatz which reads, in superfield notation,

$$\begin{aligned} L^{ij} = & \mathcal{Z} \mathcal{A} \nabla^{\alpha i} L \nabla_{\alpha}^j L + \bar{\mathcal{Z}} \bar{\mathcal{A}} \bar{\nabla}_{\dot{\alpha}}^i L \bar{\nabla}^{\dot{\alpha} j} L + \mathcal{Z} \mathcal{B}_I \nabla^{\alpha(i} X^I \nabla_{\alpha}^{j)} L + \bar{\mathcal{Z}} \bar{\mathcal{B}}_I \bar{\nabla}_{\dot{\alpha}}^{(i} \bar{X}^I \bar{\nabla}^{\dot{\alpha} j)} L \\ & + \mathcal{C}_{IJ} \nabla^{\alpha i} X^I \nabla_{\alpha}^j X^J + \bar{\mathcal{C}}_{IJ} \bar{\nabla}_{\dot{\alpha}}^i \bar{X}^I \bar{\nabla}^{\dot{\alpha} j} \bar{X}^J - \frac{1}{2} \mathcal{G}_I \nabla^{ij} X^I . \end{aligned} \quad (5.26)$$

The index I runs $0, 2, \dots, n$ with $X^I = (\mathcal{Z}, \mathcal{W}^A)$. The coefficient functions $\mathcal{A}, \mathcal{B}_I, \mathcal{C}_{IJ}$ and \mathcal{G}_I fell into three distinct cases. The first case was given by

$$\begin{aligned} \mathcal{A} = & \eta_{11}(L + i\zeta) - \frac{1}{2}g , \quad \mathcal{B}_I = -\frac{1}{2}(L + i\zeta)\partial_I g - 2i\partial_I b , \quad \mathcal{C}_{IJ} = -\frac{i}{2}(L + i\zeta)\partial_I \partial_J(Zb) , \\ \mathcal{G}_I = & \text{Re} \left(\left[\frac{1}{3}\eta_{11}(L + i\zeta)^3 - \frac{i}{2}\zeta(L + i\zeta)g \right] \delta_I^0 + \frac{1}{2}(L + i\zeta)Z\partial_I(gL + 4ib) \right) , \end{aligned} \quad (5.27)$$

where the parameter ζ was defined as

$$\zeta = \frac{\text{Im}(Lg + 4ib)}{2\eta_{11}L - \text{Re}g} . \quad (5.28)$$

This led to general couplings of the VT multiplet to conformal supergravity and the abelian multiplets \mathcal{W}^A .

The second case turned out to give a total derivative and corresponded to

$$\begin{aligned} \mathcal{A} = & i\eta_{11}\zeta' - i\alpha , \quad \mathcal{B}_I = -\frac{i}{2}\zeta'\partial_I g - 2i\partial_I \gamma , \quad \mathcal{C}_{IJ} = \frac{1}{2}\zeta'\partial_I \partial_J(Zb) , \\ \mathcal{G}_I = & \text{Re} \left(2iZL\partial_I \gamma + \frac{i}{2}\zeta'ZL\partial_I g - 2\zeta'\partial_I(Zb) \right) , \\ \gamma = & \frac{i}{4}\alpha_A \mathcal{W}^A / Z , \quad \zeta' = \frac{2\alpha L + 4\text{Re}\gamma}{2\eta_{11}L - \text{Re}g} , \end{aligned} \quad (5.29)$$

where the parameters α and α_A were arbitrary real numbers. This solution can be included into the first solution by the redefinitions

$$g \rightarrow g + 2i\alpha , \quad b \rightarrow b + \gamma . \quad (5.30)$$

The third case was

$$\mathcal{A} = \mathcal{B}_I = 0 , \quad \mathcal{C}_{IJ} = -\frac{i}{8}\partial_I \partial_J(f(X)/\mathcal{Z}) , \quad \mathcal{G}_I = -\frac{1}{2}\text{Im}\partial_I(f(X)/\mathcal{Z}) , \quad (5.31)$$

where $f(X)$ is a holomorphic function of $X^I = (\mathcal{Z}, \mathcal{W}^A)$ of degree 2; this corresponded to an alternative formulation for general vector multiplet couplings.

Each of these tensor multiplets can be rewritten in a way which makes their properties somewhat more transparent. Let us begin with the first case. Making use of the constraint obeyed by L , one can rewrite L^{ij} as

$$L^{ij} = \nabla^{ij}(\mathcal{Z}\Gamma) + \bar{\nabla}^{ij}(\bar{\mathcal{Z}}\bar{\Gamma}) - \frac{1}{2} \left(\frac{\mathcal{W}^A}{\mathcal{Z}} + \frac{\bar{\mathcal{W}}^A}{\bar{\mathcal{Z}}} \right) \left(\frac{1}{2}\eta_{1A}G^{11ij} + 2\eta_{AB}G^{1Bij} \right) , \quad (5.32)$$

where G^{IJij} is given by (5.17) and we have defined

$$\Gamma = \frac{1}{6}\eta_{11}L^3 + \frac{1}{4}\eta_{1A}Y^AL^2 + \frac{1}{2}\eta_{AB}Y^AY^BL. \quad (5.33)$$

The remarkable feature of this action is that it is completely *tri-linear* in the fields L , \mathcal{W}^A , and $\bar{\mathcal{W}}^A$. Moreover, except for the single factor $\mathcal{W}^A/\mathcal{Z} + \bar{\mathcal{W}}^A/\bar{\mathcal{Z}}$ appearing in the third term, the action depends only on the combination Y^A .

The second action can be rewritten in a similar way as

$$L^{ij} = -\alpha G^{11ij} + \alpha_A G^{1Aij}. \quad (5.34)$$

Note that the action is purely *bilinear* in the fields L , \mathcal{W}^A and $\bar{\mathcal{W}}^A$. It is immediately apparent that a term of the form (5.34) arises by making the replacement $\mathcal{W}^A \rightarrow \mathcal{W}^A + m^A \mathcal{Z}$ for a real constant m^A in (5.32), consistent with the shift symmetry (5.30). From a superspace perspective, it is unclear why (5.34) should lead to a total derivative.

Finally, the last case of Lagrangian can be rewritten

$$L^{ij} = -\frac{i}{8}\nabla^{ij}\left(\frac{f(X)}{\mathcal{Z}}\right) + \text{c.c.} \quad (5.35)$$

In this case, L^{ij} is invariant under the central charge and we may utilize the projective superspace action with $\mathcal{L}^{++} = \mathcal{V}L^{++}$ (where \mathcal{V} is the prepotential for the central charge multiplet \mathcal{Z}) to rewrite the action as [58]

$$S = \frac{i}{2}\int d^4x d^4\theta \mathcal{E} f(X) + \text{c.c.} \quad (5.36)$$

Using these superfield reformulations, it is possible to show that the symmetry of the constraints generated by the redefinitions (5.19) and (5.20) can be lifted to symmetries of the action itself. Combining the first case (5.32) and the third case with $f(X) = f_{ABC}\mathcal{W}^A\mathcal{W}^B\mathcal{W}^C$ for real f_{ABC} leads to a linear multiplet

$$\begin{aligned} L^{ij} &= \nabla^{ij}(\mathcal{Z}\Gamma) + \bar{\nabla}^{ij}(\bar{\mathcal{Z}}\Gamma) \\ &\quad - \frac{1}{2}\left(\frac{\mathcal{W}^A}{\mathcal{Z}} + \frac{\bar{\mathcal{W}}^A}{\bar{\mathcal{Z}}}\right)\left(\frac{1}{2}\eta_{1A}G^{11ij} + 2\eta_{AB}G^{1Bij} - 3f_{ABC}G^{BCij}\right), \\ \Gamma &= \frac{1}{6}\eta_{11}L^3 + \frac{1}{4}\eta_{1A}Y^AL^2 + \frac{1}{2}\eta_{AB}Y^AY^BL - \frac{1}{2}f_{ABC}Y^AY^BY^C, \end{aligned} \quad (5.37)$$

which is symmetric under the shifts (5.19) and (5.20) along with

$$f_{ABC} = f'_{ABC} + \frac{1}{3}\eta'_{11}c_{ACBCC} - \frac{1}{2}\eta'_{1(A}c_{BCC)} + \eta'_{(AB}c_{C)}. \quad (5.38)$$

One can straightforwardly prove that Γ is invariant while the term involving G^{IJij} is invariant when one uses the constraint (5.16).

5.4 Comparison with existing superspace approaches

Recently, two superspace formulations of the VT multiplet, the linear and nonlinear VT multiplet, were presented in AdS and in supergravity [28]. Both of these cases correspond to special choices for the coefficients η_{IJ} , which we can now explicitly demonstrate. The nonlinear VT multiplet obeys the constraint²⁵

$$\nabla^{ij}L = -\frac{2}{\mathcal{Z}}\nabla^{\alpha(i}\mathcal{Z}\nabla_{\alpha}^{j)}L + \frac{1}{\mathcal{Z}L}\left(-\mathcal{Z}\nabla^{\alpha i}L\nabla_{\alpha}^jL + \bar{\mathcal{Z}}\bar{\nabla}_{\dot{\alpha}}^iL\bar{\nabla}^{\dot{\alpha}j}L - \frac{L^2}{2}\nabla^{ij}\mathcal{Z}\right) \quad (5.39)$$

as well as the constraint (5.7). Similarly, the Lagrangian L^{ij} constructed in [28] is simply

$$L^{ij} = \frac{1}{24}\nabla^{ij}(\mathcal{Z}L^3) + \frac{1}{24}\bar{\nabla}^{ij}(\bar{\mathcal{Z}}L^3) . \quad (5.40)$$

It is straightforward to compare these equations to (5.8) and (5.32) and observe agreement (up to an overall normalization of the Lagrangian) for the case $\eta_{1A} = \eta_{AB} = 0$.

Similarly, the linear VT multiplet, which couples not only to the central charge multiplet \mathcal{Z} but also an additional vector multiplet \mathcal{W} , obeys (5.7) along with

$$\begin{aligned} \nabla^{ij}L = & \frac{2\bar{\mathcal{W}}}{\bar{\mathcal{Z}}\mathcal{W} - \mathcal{Z}\bar{\mathcal{W}}}\left(\nabla^{\alpha(i}\mathcal{Z}\nabla_{\alpha}^{j)}L + \bar{\nabla}_{\dot{\alpha}}^{(i}\bar{\mathcal{Z}}\bar{\nabla}^{\dot{\alpha}j)}L + \frac{1}{2}L\nabla^{ij}\mathcal{Z}\right) \\ & - \frac{2\bar{\mathcal{Z}}}{\bar{\mathcal{Z}}\mathcal{W} - \mathcal{Z}\bar{\mathcal{W}}}\left(\nabla^{\alpha(i}\mathcal{W}\nabla_{\alpha}^{j)}L + \bar{\nabla}_{\dot{\alpha}}^{(i}\bar{\mathcal{W}}\bar{\nabla}^{\dot{\alpha}j)}L + \frac{1}{2}L\nabla^{ij}\mathcal{W}\right) . \end{aligned} \quad (5.41)$$

This constraint matches (5.8) for the case $\eta_{11} = \eta_{AB} = 0$. Similarly, the Lagrangian L^{ij} for this case, eq. (6.7) in [28], can be shown to match (5.32) up to an overall constant.

It should be mentioned that the nonlinear case with Chern-Simons terms (*i.e.* setting only $\eta_{1A} = 0$), which was also considered in [28], is equivalent to the most general case by a simple superfield redefinition. Given a VT multiplet with $\eta_{1A} = 0$, one can redefine L by

$$L = L' + c_A Y^A . \quad (5.42)$$

This leads to a new set of η coefficients

$$\eta'_{11} = \eta_{11} , \quad \eta'_{1A} = 2\eta_{11}c_A , \quad \eta'_{AB} = \eta_{AB} + \eta_{11}c_Ac_B . \quad (5.43)$$

The choice of c_A then sets the new value η'_{1A} , and the most general case is recovered.

6 Discussion

In this paper, we have studied component reductions from superspace using the newly-constructed conformal superspace [14] as well as the conventional formulation based on SU(2) superspace [34]. In section 3, we demonstrated how to construct component actions of vector multiplets coupled to supergravity, both for the superconformal case (which reproduces results originally constructed using the superconformal tensor calculus) and for the

²⁵We have rewritten the results of [28] to use the conformal superspace geometry of [14].

non-superconformal case. We also briefly discussed the component construction of tensor multiplet self-couplings.

For these applications, only conventional $\mathcal{N} = 2$ superspaces were used, but off-shell hypermultiplets (and even the most natural formulation of tensor multiplets) require projective superspace. For this reason, we demonstrated in section 4 explicitly how to perform component reductions in the local projective superspace formulated in [10]. Our action was a relatively simple one, but we were able to demonstrate explicitly how the auxiliary contour integral selects out the physical degrees of freedom for a simple class of models involving (possibly composite) vector and tensor multiplets. To our knowledge, this is the first direct application of the local projective superspace component reduction rule derived in [58].

It would be interesting to calculate the component action involving general two-derivative self-couplings of arctic multiplets. It is believed that the elimination of the infinite number of auxiliary fields must lead to a model of on-shell hypermultiplets coupled to conformal supergravity with a target space geometry of a hyperkähler cone. However, this has never been demonstrated by an explicit calculation. In analogy to flat projective superspace, one might construct an action in $\mathcal{N} = 1$ superspace where the arctic multiplet is decomposed as an infinite number of $\mathcal{N} = 1$ superfields. Algebraic elimination of the auxiliaries followed by a duality transformation would then lead to a model involving purely $\mathcal{N} = 1$ chiral superfields. The resulting $\mathcal{N} = 1$ action would exhibit one manifest supersymmetry as well as the requisite target space geometry and could be reduced completely to components. However, there is currently no way of handling the supergravity couplings in such an approach. It is possible that this action could instead be handled by direct reduction to components (as we have done for a far simpler model). Whether the auxiliary fields can be eliminated (even formally) in this approach remains to be seen.

In section 5, we considered an inverse procedure. Rather than deriving component results from superspace, we showed how superspace results may be obtained from components. Taking a particularly intricate component multiplet – the vector-tensor multiplet constructed by Claus *et al.* – we showed how a superfield formulation can simplify its constraint structure.²⁶ We found that the nonlinear constraints can be understood using the three relatively simple relations (5.7), (5.13) and (5.16). This last constraint in particular hints that the off-shell VT multiplet considered by Claus *et al.* could have a five-dimensional origin, as shown in flat superspace by Kuzenko and Linch for the nonlinear VT multiplet [64].

A number of unanswered questions remain regarding the VT multiplet. The most prevalent question is whether it can be derived from five dimensions when coupled to conformal supergravity. Recently, a detailed connection between off-shell conformal supergravity in 5D and 4D has been developed [66]. It would be interesting to see if the VT multiplet can indeed be derived directly in this formalism.

Secondly, the superspace geometry of the 1-form and 2-form of the VT multiplet have

²⁶The authors of [27] indeed pointed out the “need for a suitable superspace formulation” due to the complexity of their component results.

not yet been constructed in supergravity. As reviewed earlier, both the vector and tensor multiplets can be naturally encoded, respectively, in 1-form and 2-form superspace geometries with suitable superspace constraints. The VT multiplet should also have an elegant reformulation involving coupled superforms. In fact, the geometry for the linear VT multiplet in Minkowski superspace has already been worked out in a number of references [22–24]. Generalizing these results to supergravity and to the more general VT multiplet considered here would be of interest.

Another interesting question is whether the off-shell VT multiplet constructed in [27] can be generalized. In 2001, a new off-shell formulation of the VT multiplet in flat superspace was constructed by Theis [30] (see also [29]). The novel feature of this formulation is that the VT multiplet *itself* provides the vector field necessary for gauging the central charge. It would be interesting to understand if such a VT multiplet can be formulated in supergravity, and what modifications are necessary to the component procedure of Claus *et al.* to accommodate this. On a related note, there exists a geometric approach to the VT multiplet in supergravity constructed using Free Differential Algebra [31, 32], which seems to be quite general. It would be interesting to understand explicitly how the off-shell formulations in superspace and in tensor calculus relate to this approach.

Although there is much to be learned about the VT multiplet, there is one immediate application for this formulation: the construction of higher derivative actions. It was demonstrated in [53] that higher derivative actions involving vector and tensor multiplets may be constructed based on the action (3.55) by treating one or both of the multiplets as composite. Building composite vector multiplets out of composite tensor multiplets naturally leads to a hierarchy of higher derivative actions. It was argued in [28] that this same principle can naturally be applied to higher derivative actions involving the VT multiplet, since there is no requirement that the multiplets \mathcal{W}^A be *fundamental*; we can just as easily consider them to be composite and built out of other vector multiplets or tensor multiplets along the lines discussed in [53]. This quite economically accommodates higher derivative interactions into the VT Lagrangian. It would be interesting to see if such interactions can be found within string calculations which inspired the renewal of interest in the VT multiplet.

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A Notations and conventions

Our notations and conventions follow mostly those in [36]. We briefly summarize them here.

We use two-component notation where dotted and undotted spinor indices are raised and lowered by ε tensors

$$\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta, \quad \bar{\chi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}}, \quad (\text{A.1})$$

obeying

$$\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}, \quad \varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma, \quad \varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\alpha}}^{\dot{\gamma}}, \quad \varepsilon^{12} = 1.$$

Similarly SU(2) indices are raised and lowered by ε_{ij} and ε^{ij} having the same properties as $\varepsilon_{\alpha\beta}$. Spinor indices are contracted as

$$\psi\chi := \psi^\alpha\chi_\alpha, \quad \bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}. \quad (\text{A.2})$$

For spinors which are also isospinors, we define

$$\psi\chi = \psi_i^\alpha\chi_\alpha^i, \quad \bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}^i\bar{\chi}_i^{\dot{\alpha}}. \quad (\text{A.3})$$

The metric is $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$. The sigma matrices are defined as

$$(\sigma^a)_{\alpha\dot{\alpha}} = (1, \vec{\sigma}), \quad (\tilde{\sigma}^a)^{\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}(\sigma^a)_{\beta\dot{\beta}} = (1, -\vec{\sigma}) \quad (\text{A.4})$$

and have the properties

$$(\sigma_a)_{\alpha\dot{\beta}}(\tilde{\sigma}_b)^{\dot{\beta}\beta} = -\eta_{ab}\delta_\alpha^\beta - 2(\sigma_{ab})_\alpha^\beta, \quad (\text{A.5})$$

$$(\tilde{\sigma}_a)^{\dot{\alpha}\beta}(\sigma_b)_{\beta\dot{\beta}} = -\eta_{ab}\delta_{\dot{\beta}}^{\dot{\alpha}} - 2(\tilde{\sigma}_{ab})_{\dot{\beta}}^{\dot{\alpha}}, \quad (\text{A.6})$$

together with the following useful identities

$$\begin{aligned} (\sigma^a)_{\alpha\dot{\alpha}}(\sigma_a)_{\beta\dot{\beta}} &= -2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}, \\ (\sigma_{ab})_{\alpha\beta}(\sigma^{ab})_{\gamma\delta} &= -2\varepsilon_{\gamma(\alpha}\varepsilon_{\beta)\delta}, \\ (\sigma_{ab})_{\alpha\beta}(\tilde{\sigma}^{ab})_{\dot{\gamma}\dot{\delta}} &= 0, \\ \text{tr}(\sigma_{ab}\sigma_{cd}) &= (\sigma_{ab})_\alpha^\beta(\sigma_{cd})_\beta^\alpha = -\eta_{a[c}\eta_{d]b} - \frac{i}{2}\varepsilon_{abcd}, \\ (\sigma_{[a})_{\alpha\dot{\beta}}(\sigma_{bc]})_{\gamma\delta} &= \frac{i}{3}\varepsilon_{abcd}\varepsilon_{\alpha(\gamma}(\sigma^d)_{\delta)\dot{\beta}}, \\ \varepsilon^{abcd}\varepsilon_{a'b'c'd'} &= -4!\delta_{[a'}^a\delta_{b'}^b\delta_{c'}^c\delta_{d']}^d, \\ (\sigma_{ab}\sigma_c)_\alpha^{\dot{\alpha}} &= (\sigma_{ab})_\alpha^\beta(\sigma_c)_\beta^{\dot{\alpha}} = -\eta_{c[a}(\sigma_{b]})_\alpha^{\dot{\alpha}} - \frac{i}{2}\varepsilon_{abcd}(\sigma^d)_\alpha^{\dot{\alpha}}, \\ \varepsilon_{abcd}(\sigma^{cd})_{\alpha\beta} &= -2i(\sigma_{ab})_{\alpha\beta}, \quad \varepsilon_{abcd}(\tilde{\sigma}^{cd})_{\dot{\alpha}\dot{\beta}} = 2i(\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}}. \end{aligned}$$

The antisymmetric tensor is

$$\varepsilon^{0123} = -\varepsilon_{0123} = 1, \quad (\text{A.8})$$

and (anti-)symmetrization includes a normalization factor, for example

$$V_{[ab]} = \frac{1}{2!}(V_{ab} - V_{ba}) , \quad \psi_{(\alpha\beta)} = \frac{1}{2!}(\psi_{\alpha\beta} + \psi_{\beta\alpha}) . \quad (\text{A.9})$$

For superform indices, we introduce graded antisymmetrization, *e.g.*

$$V_{\{AB\}} = \frac{1}{2!}(V_{AB} - (-)^{ab}V_{BA}) . \quad (\text{A.10})$$

When an index is not included, we separate it with vertical bars, *e.g.*

$$\begin{aligned} T_{[AB}{}^D F_{D|C\}} = \frac{1}{3!} & \left(T_{AB}{}^D F_{DC} - (-)^{ab} T_{BA}{}^D F_{DC} + (-)^{ca+cb} T_{CA}{}^D F_{DB} \right. \\ & \left. - (-)^{cb} T_{AC}{}^D F_{DB} + (-)^{ab+ac} T_{BC}{}^D F_{DA} - (-)^{ab+ac+cb} T_{CB}{}^D F_{DA} \right) . \end{aligned} \quad (\text{A.11})$$

A vector V_a can be rewritten with spinor indices as

$$V_{\alpha\dot{\beta}} = (\sigma^a)_{\alpha\dot{\beta}} V_a , \quad V_a = -\frac{1}{2}(\tilde{\sigma}_a)^{\dot{\beta}\alpha} V_{\alpha\dot{\beta}} . \quad (\text{A.12})$$

A real antisymmetric tensor, $F_{ab} = -F_{ba}$ is converted to spinor indices as

$$F_{\alpha\beta} = \frac{1}{2}(\sigma^{ab})_{\alpha\beta} F_{ab} , \quad \bar{F}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}(\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}} F_{ab} , \quad F_{ab} = (\sigma_{ab})^{\alpha\beta} F_{\alpha\beta} - (\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}} \bar{F}^{\dot{\alpha}\dot{\beta}} . \quad (\text{A.13})$$

B Solution to constraints for the tensor multiplet

In this section we summarize the analysis for solving the Bianchi identities associated with the tensor multiplet field strength in the supergravity formulation of [14].

The field strength,

$$H = dB \implies H_{ABC} = 3(\nabla_{[A} B_{BC\}} - T_{[AB}{}^D B_{D|C\}}) , \quad (\text{B.1})$$

satisfies the Bianchi identity

$$\nabla_{[A} H_{BCD\}} - \frac{3}{2} T_{[AB}{}^E H_{E|CD\}} = 0 . \quad (\text{B.2})$$

We impose the mass dimension- $\frac{3}{2}$ constraints

$$H_{\alpha\beta\gamma}^{i\ j\ k} = H_{i\ j\ k}^{\dot{\alpha}\dot{\beta}\dot{\gamma}} = H_{\alpha\beta k}^{i\ j\ \dot{\gamma}} = H_{i\ j\ \gamma}^{\dot{\alpha}\dot{\beta} k} = 0 \quad (\text{B.3})$$

and proceed to analyze the consequences of the Bianchi identities at each mass dimension.

Mass dimension-2

At mass dimension 2, we must analyze two independent cases. Putting $A = \frac{i}{\alpha}$, $B = \frac{j}{\beta}$, $C = \frac{k}{\gamma}$, $D = \frac{\dot{\delta}}{l}$ gives

$$\delta_l^i H_{\alpha\dot{\beta}\gamma}^{j\ k} + \delta_l^j H_{\beta\dot{\alpha}\gamma}^{i\ k} + \delta_l^k H_{\gamma\dot{\delta}\alpha\beta}^{i\ j} = 0 . \quad (\text{B.4})$$

One can easily check that the only solution to this equation is $H_{\alpha\dot{\alpha}\beta\gamma}^{i\ j} = 0$. Similarly $H_{\alpha\dot{\alpha}i\ k}^{\dot{\beta}\dot{\gamma}} = 0$.

Putting $A = \overset{i}{\alpha}$, $B = \overset{j}{\beta}$, $C = \overset{\dot{\alpha}}{k}$, $D = \overset{\dot{\beta}}{l}$ gives

$$\delta_j^i H_{\alpha\dot{\alpha}\beta\dot{\beta}l}^k + \delta_l^k H_{\beta\dot{\beta}\alpha\dot{\alpha}j}^i + \delta_l^i H_{\alpha\dot{\alpha}\beta\dot{\beta}j}^k + \delta_j^k H_{\beta\dot{\beta}\alpha\dot{\alpha}l}^i = 0 . \quad (\text{B.5})$$

This is solved by

$$H_{\alpha\dot{\alpha}\beta\dot{\beta}l}^k = -4\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\mathcal{G}_l^k + \delta_l^k(\varepsilon_{\dot{\alpha}\dot{\beta}}H_{\alpha\beta} + \varepsilon_{\alpha\beta}H_{\dot{\alpha}\dot{\beta}}) , \quad (\text{B.6})$$

where \mathcal{G}_{ij} is a *traceless hermitian* superfield of vanishing U(1) weight and we make the identifications

$$\mathcal{G}^{ij} = -\frac{1}{16}H^{\alpha\dot{\alpha}}_{\alpha\dot{\alpha}}{}^{(ij)} = -\frac{1}{16}H^{\alpha\dot{\alpha}}{}^i{}_j{}^{\dot{\alpha}\dot{\alpha}} , \quad H_{\alpha\beta} = -\frac{1}{4}H_{(\alpha}{}^{\dot{\gamma}}{}_{\beta)}{}^{\dot{\gamma}}{}^i{}_i , \quad H_{\dot{\alpha}\dot{\beta}} = -\frac{1}{4}H_{\alpha(\dot{\alpha}}{}^{\alpha i}{}_{\dot{\beta})i} . \quad (\text{B.7})$$

Now from the definition of H we see that $H_{a\alpha i}^{\dot{\beta}}$ contains a term proportional to $(\sigma^b)_\alpha{}^{\dot{\beta}}B_{ab}$ and thus by redefining B_{ab} we can make $H_{\alpha\beta} = 0$ (similarly we impose $H_{\dot{\alpha}\dot{\beta}} = 0$). Hence we end up with the *constraints*:

$$H_{\alpha\beta\gamma}^{ijk} = H_i^{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}_{jk} = H_{\alpha\beta k}^{ij\dot{\gamma}} = H_i^{\dot{\alpha}\dot{\beta}k}{}_{j\gamma} = 0 , \quad (\text{B.8a})$$

$$H_{a\alpha\beta}^{ij} = 0 , \quad H_{a i}^{\dot{\alpha}\dot{\beta}}{}_{j\dot{\gamma}} = 0 , \quad H_{a\alpha j}^{i\dot{\alpha}}{}_{\dot{\gamma}} = 2(\sigma_a)_\alpha{}^{\dot{\alpha}}\mathcal{G}_{j\dot{\gamma}} . \quad (\text{B.8b})$$

Mass dimension- $\frac{5}{2}$:

Putting $A = a$, $B = \overset{j}{\beta}$, $C = \overset{k}{\gamma}$ and $D = \overset{l}{\delta}$ yields no additional constraints.

Putting $A = a$, $B = \overset{j}{\beta}$, $C = \overset{k}{\gamma}$ and $D = \overset{\dot{\delta}}{l}$ gives

$$(\sigma_a)_\gamma{}^{\dot{\delta}}\nabla_\beta^j\mathcal{G}_l^k + (\sigma_a)_\beta{}^{\dot{\delta}}\nabla_\gamma^k\mathcal{G}_l^j + i\delta_l^j(\sigma^b)_\beta{}^{\dot{\delta}}H_{ab\gamma}^k + i\delta_l^k(\sigma^b)_\gamma{}^{\dot{\delta}}H_{ab\beta}^j = 0 . \quad (\text{B.9})$$

Converting to spinor indices

$$-2\varepsilon_{\alpha\gamma}\varepsilon_{\dot{\alpha}\dot{\beta}}\nabla_\beta^j\mathcal{G}_l^k - 2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\nabla_\gamma^k\mathcal{G}_l^j + i\delta_l^jH_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma}^k + i\delta_l^kH_{\alpha\dot{\alpha}\gamma\dot{\beta}\beta}^j = 0 . \quad (\text{B.10})$$

Taking the symmetric part in j, k and l gives

$$\nabla_\alpha^{(i}\mathcal{G}^{jk)} = 0 . \quad (\text{B.11})$$

Alternatively contracting k and l gives

$$-2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\nabla_\gamma^k\mathcal{G}_k^j + iH_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma}^j + 2iH_{\alpha\dot{\alpha}\gamma\dot{\beta}\beta}^j = 0 . \quad (\text{B.12})$$

Now taking the symmetric part in $\dot{\alpha}$ and $\dot{\beta}$ we deduce $H_{\alpha(\dot{\alpha}}{}^{\beta}{}_{\dot{\beta})}{}^j{}_\gamma = 0$. On the other hand contracting $\dot{\alpha}$ and $\dot{\beta}$ and taking the symmetric and antisymmetric parts in β and γ gives

$$H_{\alpha\dot{\alpha}(\beta}{}^{\dot{\alpha}}{}_{\gamma)j} = \frac{4i}{3}\varepsilon_{\alpha(\beta}\nabla_{\gamma)}^k\mathcal{G}_{kj} , \quad H_{\alpha\dot{\alpha}[\beta}{}^{\dot{\alpha}}{}_{\gamma]j} = 2i\varepsilon_{\beta\gamma}\nabla_\alpha^k\mathcal{G}_{kj} . \quad (\text{B.13})$$

Then using the fact that $H_{ab\gamma}^k = H_{[ab]\gamma}^k$ and the above results we find

$$H_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma}^j = \frac{4i}{3}\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon_{\gamma(\alpha}\nabla_{\beta)}^k\mathcal{G}_{kj}^j , \quad \nabla_\alpha^{(i}\mathcal{G}^{jk)} = 0 . \quad (\text{B.14})$$

Similarly we deduce

$$H_{\alpha\dot{\alpha}\beta\dot{\beta}j} = \frac{4i}{3}\varepsilon_{\alpha\beta}\delta_{(\dot{\alpha}}^{\dot{\gamma}}\bar{\nabla}_{\dot{\beta})k}\mathcal{G}^k{}_j, \quad \bar{\nabla}_{\dot{\alpha}}^{(i}\mathcal{G}^{jk)} = 0. \quad (\text{B.15})$$

Mass dimension-3:

Putting $A = a$, $B = b$, $C = \dot{i}_\alpha$, $D = \dot{\alpha}_j$ gives

$$-2i(\sigma^c)_\alpha{}^{\dot{\alpha}}\delta_j^i H_{abc} = -4(\sigma_{[a})_\alpha{}^{\dot{\alpha}}\nabla_{b]}\mathcal{G}^i{}_j + \frac{2i}{3}(\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}}\nabla_\alpha^i\bar{\nabla}_k^{\dot{\beta}}\mathcal{G}^k{}_j + \frac{2i}{3}(\sigma_{ab})_\alpha{}^\beta\bar{\nabla}_j^{\dot{\alpha}}\nabla_\beta^k\mathcal{G}^i{}_k. \quad (\text{B.16})$$

Now contracting i and j gives

$$-4i(\sigma^c)_\alpha{}^{\dot{\alpha}}H_{abc} = \frac{2i}{3}(\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}}\nabla_\alpha^j\bar{\nabla}_k^{\dot{\beta}}\mathcal{G}^k{}_j + \frac{2i}{3}(\sigma_{ab})_\alpha{}^\beta\bar{\nabla}_k^{\dot{\alpha}}\nabla_\beta^j\mathcal{G}^k{}_j. \quad (\text{B.17})$$

Then rearranging and simplifying the above gives

$$H_{abc} = \frac{i}{24}\varepsilon_{abcd}(\sigma^d)_\alpha{}^{\dot{\beta}}[\nabla_\alpha^j, \bar{\nabla}_k^{\dot{\beta}}]\mathcal{G}^k{}_j \equiv \varepsilon_{abcd}\tilde{H}^d. \quad (\text{B.18})$$

All other Bianchi identities yield no further constraints, however the following useful identities can be found:

$$\begin{aligned} \nabla_\alpha^i\tilde{H}^a &= \frac{2}{3}(\sigma^{ab})_\alpha{}^\beta\nabla_b\nabla_\beta^k\mathcal{G}^i{}_k + \frac{2}{3}(\tilde{\sigma}^{ab})^{\dot{\beta}}{}_{\dot{\gamma}}T_{b\alpha\dot{\beta}}^i\nabla_k^{\dot{\gamma}}\mathcal{G}^k{}_j + \varepsilon^{abcd}(\sigma_b)_\alpha{}^{\dot{\beta}}T_{cd\dot{\beta}}^j\mathcal{G}^i{}_j, \\ \nabla_a\tilde{H}^a &= -\frac{2}{3}T_{\alpha\beta i}^\alpha\nabla^{\beta k}\mathcal{G}^i{}_k + \text{c.c.} \end{aligned} \quad (\text{B.19})$$

Having solved the constraints for the three-form, we now turn to finding a solution for the gauge two-form under some appropriate constraints. We make use of the Yang-Mills constraints

$$B_{\alpha\beta}^{ij} = 4i\varepsilon_{\alpha\beta}\varepsilon^{ij}\bar{\Psi}, \quad B_i^{\dot{\alpha}\dot{\beta}} = 4i\varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon_{ij}\Psi, \quad B_{\alpha\dot{\beta}}^{ij} = 0. \quad (\text{B.20})$$

We can then solve (2.22) for B under these conditions. We again proceed to analyze the consequences by mass dimension.

Mass dimension- $\frac{3}{2}$:

Letting $A = \dot{i}_\alpha$, $B = \dot{j}_\beta$ and $C = \dot{k}_\gamma$ gives

$$\nabla_\alpha^i B_{\beta\gamma}^{jk} + \nabla_\beta^j B_{\alpha\gamma}^{ik} + \nabla_\gamma^k B_{\alpha\beta}^{ij} = 0. \quad (\text{B.21})$$

Substituting $B_{\alpha\beta}^{ij} = 4i\varepsilon_{\alpha\beta}\varepsilon^{ij}\bar{\Psi}$ and contracting β with γ and j with k gives

$$\nabla_\alpha^i\bar{\Psi} = 0. \quad (\text{B.22})$$

Similarly we also have $\bar{\nabla}_i^{\dot{\alpha}}\Psi = 0$.

Now letting $A = \dot{i}_\alpha$, $B = \dot{j}_\beta$ and $C = \dot{k}_k$ gives

$$\nabla_\alpha^i B_{\beta k}^{j\dot{\alpha}} + \nabla_\beta^j B_{\alpha k}^{i\dot{\alpha}} + \bar{\nabla}_k^{\dot{\alpha}} B_{\alpha\beta}^{ij} + 2i\delta_k^i(\sigma^a)_\alpha{}^{\dot{\alpha}}B_{a\beta}^j + 2i\delta_k^j(\sigma^a)_\beta{}^{\dot{\alpha}}B_{a\alpha}^i = 0. \quad (\text{B.23})$$

Substituting our constraints leads to

$$4i\varepsilon_{\alpha\beta}\varepsilon^{ij}\bar{\nabla}_{\dot{\alpha}k}\bar{\Psi} + 2i\delta_k^i B_{\alpha\dot{\alpha}\beta}^j + 2i\delta_k^j B_{\beta\dot{\alpha}\alpha}^i = 0 . \quad (\text{B.24})$$

Contracting j and k

$$4i\varepsilon_{\alpha\beta}\bar{\nabla}_{\dot{\alpha}}^i\bar{\Psi} + 2iB_{\alpha\dot{\alpha}\beta}^i + 4iB_{\beta\dot{\alpha}\alpha}^i = 0 , \quad (\text{B.25})$$

and taking the symmetric part in α and β gives $B_{(\alpha}{}^{\dot{\alpha}}{}_{\beta)i} = 0$. On the other hand taking the antisymmetric part in α and β gives

$$B_{[\alpha}{}^{\dot{\alpha}}{}_{\beta]i} = 2\varepsilon_{\alpha\beta}\bar{\nabla}_{\dot{i}}^{\dot{\alpha}}\bar{\Psi} . \quad (\text{B.26})$$

Thus we have

$$B_{\alpha\dot{\alpha}\beta}^i = 2\varepsilon_{\alpha\beta}\bar{\nabla}_{\dot{\alpha}}^i\bar{\Psi} , \quad B_{\alpha\dot{\alpha}i}^{\dot{\beta}} = 2\delta_{\dot{\alpha}}^{\dot{\beta}}\nabla_{\alpha i}\Psi . \quad (\text{B.27})$$

Mass dimension-2:

Letting $A = a$, $B = \frac{j}{\beta}$ and $C = \frac{k}{\gamma}$ yields no new information. Setting $A = a$, $B = \frac{i}{\alpha}$ and $C = \frac{\dot{\beta}}{j}$ gives

$$2(\sigma_a)_{\alpha}{}^{\dot{\beta}}\mathcal{G}^i{}_j = \nabla_a B_{\alpha j}^{i\dot{\beta}} - \nabla_{\alpha} B_{a j}^{i\dot{\beta}} - \bar{\nabla}_j^{\dot{\beta}} B_{a\alpha}^i - 2i\delta_j^i(\sigma^b)_{\alpha}{}^{\dot{\beta}} B_{ab} - T_{a\alpha\dot{\gamma}}^{i k} B_{j k}^{\dot{\beta}\dot{\gamma}} - T_{a j k}^{\dot{\beta}\dot{\gamma}} B_{\alpha\gamma}^{i k} . \quad (\text{B.28})$$

Substituting our constraints and torsion and simplifying leads to

$$\begin{aligned} 4\eta_{ab}\mathcal{G}^{ij} &= \eta_{ab}\nabla^{ij}\Psi + \eta_{ab}\bar{\nabla}^{ij}\bar{\Psi} - \varepsilon^{ij}(\sigma_{ab})^{\alpha\beta}\nabla_{\alpha\beta}\Psi - \varepsilon^{ij}(\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}}\bar{\nabla}^{\dot{\alpha}\dot{\beta}}\bar{\Psi} + 4i\varepsilon^{ij}B_{ab} \\ &\quad + 4\varepsilon^{ij}(\sigma_{ab})^{\alpha\beta}W_{\alpha\beta}\bar{\Psi} + 4\varepsilon^{ij}(\sigma_{ab})_{\dot{\alpha}\dot{\beta}}\bar{W}^{\dot{\alpha}\dot{\beta}}\Psi . \end{aligned} \quad (\text{B.29})$$

From which it follows that

$$\mathcal{G}^{ij} = \frac{1}{4}\nabla^{ij}\Psi + \frac{1}{4}\bar{\nabla}^{ij}\bar{\Psi} , \quad (\text{B.30})$$

$$B_{ab} = -\frac{i}{4}(\sigma_{ab})^{\alpha\beta}(\nabla_{\alpha\beta}\Psi - 4W_{\alpha\beta}\bar{\Psi}) - \frac{i}{4}(\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}}(\bar{\nabla}^{\dot{\alpha}\dot{\beta}}\bar{\Psi} - 4\bar{W}^{\dot{\alpha}\dot{\beta}}\Psi) . \quad (\text{B.31})$$

All other Bianchi identities are identically satisfied.

C Component results from superspace

In this appendix, we briefly describe certain component results in the conformal superspace formulation of [14] as well as the supergravity formulation developed in [10], which was based on $\text{SU}(2)$ superspace [34].

C.1 Conformal superspace in components

The component structure of $\mathcal{N} = 2$ conformal superspace is fully described in [14]. We summarize here the results necessary for this paper, adapted to the notation of [36].

The superspace derivative ∇_a , when projected to lowest components, yields a fully supercovariant derivative, which we can write as

$$\begin{aligned} e_m{}^a\nabla_a &= \partial_m - \frac{1}{2}\psi_m{}^{\gamma}\nabla_{\gamma}^k - \frac{1}{2}\bar{\psi}_m{}^{\dot{\gamma}}\bar{\nabla}_{\dot{\gamma}}^k + \frac{1}{2}\omega_m{}^{bc}M_{bc} + \phi_m{}^{ij}J_{ij} + iA_m Y + b_m\mathbb{D} \\ &\quad + \mathfrak{f}_m{}^b K_b + \frac{1}{2}\phi_m{}^i S_i^{\alpha} + \frac{1}{2}\phi_m{}^{\dot{i}} \bar{S}_{\dot{\alpha}}^i . \end{aligned} \quad (\text{C.1})$$

Relative to the definitions used in [14], we have made several sign flips in the definitions of the connections to match the conventions of [36]. The connections ω_m^{ab} , ϕ_m^{ij} , b_m , and \mathfrak{f}_m^a differ by a sign from the corresponding objects in [14], while A_m and $\phi_{m\alpha}^i$ are the same. Similarly, the superfield $W_{\alpha\beta}$ used here differs by a sign while $\bar{W}_{\dot{\alpha}\dot{\beta}}$ is the same.

It is useful to introduce the partially degauged derivative

$$\nabla'_a = e_a{}^m \nabla'_m := e_a{}^m \left(\partial_m + \frac{1}{2} \omega_m^{ab} M_{ab} + \phi_m^{ij} J_{ij} + i A_m Y + b_m \mathbb{D} \right) \quad (\text{C.2})$$

in terms of which the supercovariant derivative can be written

$$\nabla_a| = \nabla'_a - \frac{1}{2} \psi_{a\dot{k}}^\gamma \nabla_\gamma^k| - \frac{1}{2} \bar{\psi}_{a\dot{k}}^{\dot{\gamma}} \bar{\nabla}_{\dot{\gamma}}^{\dot{k}}| + \mathfrak{f}_a{}^b K_b + \frac{1}{2} \phi_{a\alpha}^i S_\alpha^i + \frac{1}{2} \phi_{a\dot{\alpha}}^{\dot{i}} \bar{S}_{\dot{\alpha}}^{\dot{i}}. \quad (\text{C.3})$$

The connections ω_m^{ab} , \mathfrak{f}_m^a and $\phi_{m\alpha}^i$ are composite fields built out of the other components of the Weyl multiplet. The spin connection is defined in terms of the vierbein and the gravitino in the usual way.²⁷ The S -supersymmetry connection is given in two-component notation by

$$\begin{aligned} \phi_{\beta\dot{\beta}\alpha}^j &= \frac{i}{12} \bar{\Psi}_{\beta\dot{\alpha}\gamma}^{\dot{j}} \bar{\Psi}_{\dot{\beta}\alpha}^j + \frac{i}{6} \bar{\Psi}_{\alpha\dot{\beta}\gamma}^{\dot{j}} \bar{\Psi}_{\dot{\beta}\alpha}^j + \frac{i}{12} \varepsilon_{\beta\alpha} \bar{\Psi}_{\dot{\beta}}^{\gamma} \bar{\Psi}_{\gamma\dot{\alpha}}^j \\ &\quad - \frac{1}{6} \bar{W}_{\dot{\beta}}^{\dot{\gamma}} \psi_{\beta\gamma\alpha}^j - \frac{1}{3} \bar{W}_{\dot{\beta}}^{\dot{\gamma}} \psi_{\alpha\gamma\beta}^j + \frac{i}{2} \varepsilon_{\beta\alpha} \bar{\Sigma}_{\dot{\beta}}^j, \end{aligned} \quad (\text{C.4})$$

$$\begin{aligned} \bar{\phi}_{\beta\dot{\beta}\alpha\dot{j}} &= -\frac{i}{12} \Psi_{\beta\dot{\alpha}\gamma}^{\dot{j}} \Psi_{\dot{\beta}\alpha}^j - \frac{i}{6} \Psi_{\dot{\beta}\alpha}^{\dot{j}} \Psi_{\beta\gamma\dot{\alpha}}^j - \frac{i}{12} \varepsilon_{\dot{\beta}\alpha} \Psi_{\beta\dot{\alpha}}^{\dot{j}} \Psi_{\gamma\dot{\alpha}}^j \\ &\quad + \frac{1}{6} W_{\beta\dot{\gamma}}^{\gamma} \bar{\psi}_{\gamma\dot{\beta}\alpha\dot{j}} + \frac{1}{3} W_{\beta\dot{\gamma}}^{\gamma} \bar{\psi}_{\gamma\dot{\alpha}\beta\dot{j}} + \frac{i}{2} \varepsilon_{\dot{\beta}\alpha} \Sigma_{\beta\dot{j}}. \end{aligned} \quad (\text{C.5})$$

Only the trace of the special conformal connection \mathfrak{f}_m^a is required for our calculations:

$$\begin{aligned} \mathfrak{f}_a{}^a &= e_a{}^m \mathfrak{f}_m^a = -D - \frac{1}{12} \hat{\mathcal{R}} + \frac{1}{24} \varepsilon^{mnpq} (\bar{\psi}_m^j \tilde{\sigma}_n \nabla'_p \psi_{qj}) - \frac{1}{24} \varepsilon^{mnpq} (\psi_{mj} \sigma_n \nabla'_p \bar{\psi}_q^j) \\ &\quad + \frac{i}{8} (\psi_{aj} \sigma^a \bar{\Sigma}^j) - \frac{i}{8} (\bar{\psi}_a^j \tilde{\sigma}^a \Sigma_j) + \frac{1}{12} W^{ab+} (\bar{\psi}_a^j \bar{\psi}_{bj}) - \frac{1}{12} W^{ab-} (\psi_{aj} \psi_b^j). \end{aligned} \quad (\text{C.6})$$

Here $\hat{\mathcal{R}} = \hat{\mathcal{R}}(e, \omega)$ corresponds to the Poincaré version of the Lorentz curvature, constructed in the usual way from the spin connection.

C.2 SU(2) superspace in components

Now for SU(2) superspace we introduce

$$\mathcal{D}'_a = e_a{}^m \mathcal{D}_m| \equiv e_a{}^m \left(\partial_m + \frac{1}{2} \omega_m^{bc} M_{bc} + \phi_m^{ij} J_{ij} \right), \quad (\text{C.7})$$

with the algebra

$$[\mathcal{D}'_a, \mathcal{D}'_b] = \mathcal{T}_{ab}{}^c(x) \mathcal{D}'_c + \frac{1}{2} R_{ab}{}^{cd}(x) M_{cd} + R_{ab}{}^{ij}(x) J_{ij}, \quad (\text{C.8})$$

²⁷See *e.g.* [14] for the relation, which holds here except for an overall sign flip in the definition of the spin connection.

where

$$\mathcal{T}_{ab}{}^c := e_a{}^m e_b{}^n T_{mn}{}^c, \quad (\text{C.9})$$

$$\mathcal{R}_{ab}{}^{cd} := e_a{}^m e_b{}^n R_{mn}{}^{cd}, \quad (\text{C.10})$$

$$\mathcal{R}_{ab}{}^{ij} := e_a{}^m e_b{}^n R_{mn}{}^{ij}. \quad (\text{C.11})$$

Now considering the projection of the covariant derivative algebra, $[\mathcal{D}_m, \mathcal{D}_n]$ we find the following relations for the torsion and curvatures:

$$T_{mn}{}^c| = -2\mathcal{D}'_{[m}e_{n]}{}^c = -2\partial_{[m}e_{n]}{}^c + 2\omega_{[mn]}{}^c, \quad (\text{C.12})$$

$$T_{mn}{}^\gamma| = -\mathcal{D}'_{[m}\psi_{n]}{}^\gamma, \quad (\text{C.13})$$

$$R_{mn}{}^{cd}| = 2\partial_{[m}\omega_{n]}{}^{cd} + 2\omega_{[m}{}^{cb}\omega_{n]}{}^d, \quad (\text{C.14})$$

$$R_{mn}{}^{ij}| = 2\partial_{[m}\phi_{n]}{}^{ij} + 2\phi_{[m}{}^i\phi_{n]}{}^{kj}. \quad (\text{C.15})$$

The projection of their corresponding covariantized versions can then be found to be

$$T_{ab}{}^c| = 0 = \mathcal{T}_{ab}{}^c - i\psi_{[a}{}^\gamma\bar{\psi}_{b]}{}^k(\sigma^c)_\gamma{}^\delta, \quad (\text{C.16})$$

$$T_{ab}{}^\gamma| = -\Psi_{ab}{}^\gamma + 2\psi_{[a}{}^\alpha T_{b]}{}^i{}_\alpha{}^\gamma + 2\bar{\psi}_{[a}{}^i T_{b]}{}^{\dot{\alpha}}{}_\gamma{}^\gamma, \quad (\text{C.17})$$

$$\begin{aligned} R_{ab}{}^{cd}| &= \mathcal{R}_{ab}{}^{cd} + \psi_{[a}{}^\gamma R_{b]}{}^k{}_\gamma{}^{cd} + \bar{\psi}_{[a}{}^k R_{b]}{}^{\dot{\gamma}}{}_k{}^{cd} + \frac{1}{4}\psi_{[a}{}^\gamma\psi_{b]}{}^\delta R_{\gamma\delta}{}^{kl}{}^{cd} \\ &\quad + \frac{1}{4}\bar{\psi}_{[a}{}^k\bar{\psi}_{b]}{}^{\dot{\delta}} R_{k\dot{\delta}}{}^{\dot{\gamma}}{}^{\dot{\delta}}{}^{cd} + \frac{1}{2}\psi_{[a}{}^\gamma\bar{\psi}_{b]}{}^{\dot{\delta}} R_{\gamma\dot{\delta}}{}^{kl}{}^{cd}, \end{aligned} \quad (\text{C.18})$$

$$\begin{aligned} R_{ab}{}^{kl}| &= \mathcal{R}_{ab}{}^{kl} + \psi_{[a}{}^\gamma R_{b]}{}^j{}_\gamma{}^{kl} + \bar{\psi}_{[a}{}^j R_{b]}{}^{\dot{\gamma}}{}_j{}^{kl} + \frac{1}{4}\psi_{[a}{}^\gamma\psi_{b]}{}^\delta R_{\gamma\delta}{}^{ij}{}^{kl} \\ &\quad + \frac{1}{4}\bar{\psi}_{[a}{}^i\bar{\psi}_{b]}{}^{\dot{\delta}} R_{i\dot{\delta}}{}^{\dot{\gamma}}{}^{\dot{\delta}}{}^{kl} + \frac{1}{2}\psi_{[a}{}^\gamma\bar{\psi}_{b]}{}^{\dot{\delta}} R_{\gamma\dot{\delta}}{}^{ij}{}^{kl}, \end{aligned} \quad (\text{C.19})$$

where

$$\Psi_{ab}{}^\gamma := 2\mathcal{D}'_{[a}\psi_{b]}{}^\gamma - \mathcal{T}_{ab}{}^c\psi_{c}{}^\gamma, \quad \psi_a{}^\gamma := e_a{}^m\psi_{m}{}^\gamma. \quad (\text{C.20})$$

From the above results we can derive the following useful relations:

$$\mathcal{T}_{ab}{}^c = i\psi_{[a}{}^\gamma\bar{\psi}_{b]}{}^k(\sigma^c)_\gamma{}^\delta, \quad (\text{C.21})$$

$$\begin{aligned} \mathcal{D}_\gamma^k W_{\alpha\beta}| &= -\frac{2}{3}\varepsilon_{\gamma(\alpha}\mathcal{D}_{\beta)}{}^j\bar{S}^{jk}| - (\sigma^{ab})_{\alpha\beta}\psi_{ab}{}^k - 2i\varepsilon_{\gamma(\alpha}\psi_{\beta)}{}^{ak}G_a| \\ &\quad + 4i\varepsilon_{\gamma(\alpha}(\sigma^{ab})_{\beta)}{}^{\dot{\delta}}\psi_a{}^{\delta k}G_b| - i\varepsilon_{\gamma(\alpha}(\sigma^a)_{\beta)}{}^{\dot{\alpha}}\bar{\psi}_{a\dot{\alpha}}{}^{\dot{\delta}}\bar{S}^{ik}| + i(\sigma^a)_{(\alpha}{}^{\dot{\alpha}}\bar{\psi}_{a\dot{\alpha}}{}^k W_{\beta)}{}_\gamma| \\ &\quad - i\varepsilon_{\gamma(\alpha}(\sigma^a)_{\beta)}{}^{\dot{\beta}}\bar{\psi}_{a\dot{\beta}}{}^k\bar{Y}^{\dot{\alpha}\dot{\beta}}|, \end{aligned} \quad (\text{C.22})$$

$$\begin{aligned} \bar{\mathcal{D}}_k{}^{\dot{\gamma}} Y_{\alpha\beta}| &= (\sigma^{ab})_{\alpha\beta}\bar{\psi}_{ab}{}^{\dot{\gamma}} - 2i(\sigma^a)_{(\alpha}{}^{\dot{\alpha}}(\bar{\sigma}^b)^{\dot{\gamma}})_{\beta)}\bar{\psi}_{a\dot{\alpha}}{}^k G_b| \\ &\quad + i(\sigma^a)_{(\alpha}{}^{\dot{\gamma}}\psi_{a\beta)}{}^i S_{ik}| + i(\sigma^a)_{(\alpha\dot{\delta}}\psi_{a\beta)}{}^k \bar{W}^{\dot{\gamma}\dot{\delta}}| + i(\bar{\sigma}^a)^{\dot{\gamma}}{}_{(\alpha}\psi_{a\beta)}{}^{\dot{\gamma}} Y_{\beta)}{}_\gamma|. \end{aligned} \quad (\text{C.23})$$

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